MTH 265, Exam #1, Summer 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Use $\vec{u} = \langle 1,2,-4\rangle, \vec{v} = \langle 3,-2,5\rangle$ to find the following. (4 points each)
   a. $\vec{u} + \vec{v}$

   $$\langle 4,0,1\rangle$$

   b. $||\vec{v}||$

   $$\sqrt{9+4+25} = \sqrt{38}$$

   c. Write a unit vector in the direction of v

   $$\left\langle \frac{3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}} \right\rangle$$

   d. Find $\vec{u} \cdot \vec{v}$

   $$3-4-20 = -21$$

   e. Find the angle between $\vec{u}$ and $\vec{v}$ in radians, and in degrees.

   $$\cos\theta = \frac{-21}{\sqrt{21}\cdot\sqrt{38}} \qquad \theta \approx 2.4089 \text{ radians}$$
   $$\approx 138.02°$$

f. Find $\vec{u} \times \vec{v}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{vmatrix} = (10-8)\hat{i} - (5+12)\hat{j} + (-2-6)\hat{k}$$

$$\langle 2, -17, -8 \rangle$$

2. Find the volume of the parallelepiped bounded by the vectors $\vec{u} = \langle 1,2,-1 \rangle, \vec{v} = \langle 4,0,-3 \rangle$, $\vec{w} = \langle -1,3,4 \rangle$. (5 points)

$$\begin{vmatrix} 1 & 2 & -1 \\ 4 & 0 & -3 \\ -1 & 3 & 4 \end{vmatrix} = 1(0+9) - 2(\underbrace{16-3}_{13}) + (-1)(12-0) =$$

$$9 - 26 - 12 = -29$$

$$|-29| = 29$$

3. Identify the quadric surface $\frac{z^2}{16} - x^2 - \frac{y^2}{25} = 1$. (4 points)

hyperboloid of 2 sheets

4. Use polar coordinates to find the limit if it exists or determine if it does not. (6 points)

$$\lim_{(x,y)\to(0,0)} \frac{xy - 2y^2}{x^2 + y^2}$$

let $y = kx$ $\quad \lim_{x\to 0} \frac{xkx - 2(kx)^2}{x^2 + (kx)^2}$,

$$\lim_{x\to 0} \frac{kx^2 - 2k^2x^2}{x^2 + k^2x^2} =$$

$$\lim_{x\to 0} \frac{\cancel{x^2}(k - 2k^2)}{\cancel{x^2}(1 + k^2)} = \frac{k(1-2k)}{1+k^2}$$

$\Rightarrow$ DNE

$r^2 = x^2 + y^2$

let $y = r\sin\theta, x = r\cos\theta$

$$\lim_{r\to 0} \frac{r^2\cos\theta\sin\theta - 2r^2\sin^2\theta}{r^2} = \lim_{r\to 0} \cos\theta\sin\theta - 2\sin^2\theta \Rightarrow \text{DNE}$$

5. Find the limit if it exists or determine if it does not. (6 points each)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^2}$$

$x^3 = y^2$
$x^{3/2} = y$
let $y = kx^{3/2}$

$$\lim_{x\to 0} \frac{x^2 kx^{3/2}}{x^3 + (kx^{3/2})^2} = \lim_{x\to 0} \frac{kx^{7/2}}{x^3 + k^2x^3} = \lim_{x\to 0} \frac{\cancel{x^3}(kx^{1/2})}{\cancel{x^3}(1+k^2)}.$$

$$\lim_{x\to 0} \frac{k\sqrt{x}}{1+k^2} = 0$$

6. Find the value of the line integral $\int_C (x^2y^3 - \sqrt{x})ds$ on the path following the curve $y = \sqrt{x}$ between the points (0,0) and (4,2). (10 points)

$r(t) = t\hat{\imath} + t^{1/2}\hat{\jmath}$
$r'(t) = 1\hat{\imath} + \frac{1}{2\sqrt{t}}\hat{\jmath}$
$ds = \sqrt{1+\frac{1}{4t}}\,dt$

$$\int_0^4 (t^2t^{3/2} - t^{1/2})\sqrt{1+\frac{1}{4t}}\,dt$$

$$\int_0^4 (t^{7/2} - t^{1/2})\sqrt{\frac{4t+1}{4t}}\,dt = \frac{1}{2}\int_0^4 \frac{\cancel{t^{1/2}}(t^3-1)\sqrt{4t+1}}{\cancel{t^{1/2}}}\,dt$$

$$= (4t+1)^{3/2}(140t^3 - 30t^2 + 6t - 421)/5040 \Big|_0^4$$

$$\frac{421 + 137411\sqrt{17}}{5040} \approx 112.50$$

7. Find the total differential of $w = x^2yz^{1/3} + \ln yz$. Use the value of the function at the point $(1, 2, 8)$ to estimate the value of the function at $(1.2, 1.95, 8.1)$ (7 points)

$$dw = (2xyz^{1/3})\Delta x + (x^2z^{1/3} + \tfrac{1}{y})\Delta y + (\tfrac{1}{3}x^2yz^{-2/3} + \tfrac{1}{z})\Delta z$$

$$= (2\cdot1\cdot2\cdot2)(0.2) + (1^2\cdot2 + \tfrac{1}{2})(-0.05) + (\tfrac{1}{3}\cdot1\cdot2\cdot\tfrac{1}{4} + \tfrac{1}{8})(0.1)$$

$$= 8(0.2) + (\tfrac{5}{2})(-0.05) + \tfrac{7}{24}(0.1) \approx \frac{361}{240} = 1.5041\overline{6}$$

$w(1,2,8) = 1^2(2)\sqrt[3]{8} + \ln 16 = 4 + \ln 16 \approx 6.772588722$ $\quad w(1.2, 1.95, 8.1) \approx 4 + \ln 16 + 361/240 \approx 8.276755\ldots$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

8. Find the equation of the plane perpendicular to the line $\frac{x+2}{5} = y - 1 = \frac{z-6}{9}$, passing through the point $(3, -2, 8)$. (6 points)

(x_0, y_0, z_0)

$\vec{u}_{\text{line}} = \langle 5, 1, 9\rangle = \vec{n}_{\text{plane}}$

Plane: $5(x-3) + (y+2) + 9(z-8) = 0$

9. Given the quadric surface $\frac{z^2}{16} - x^2 - \frac{y^2}{25} = 1$, convert the equation to cylindrical and spherical coordinates. Use technology to graph the function in each coordinate system to verify your conversions. (6 points)

$\frac{1}{16}\rho^2\cos^2\varphi - \rho^2\sin^2\varphi\cos^2\theta - \frac{1}{25}\rho^2\sin^2\varphi\sin^2\theta = 1 \Rightarrow \rho^2(25\cos^2\varphi - 400\sin^2\varphi\cos^2\theta - 16\sin^2\varphi\sin^2\theta) = 400$

$\rho^2 = 400/(25\cos^2\varphi - 400\sin^2\varphi\cos^2\theta - 16\sin^2\varphi\sin^2\theta)$ spherical

$25z^2 = 400 + 400x^2 + 16y^2 \Rightarrow 25z^2 = 36 + 400r^2\cos^2\theta + 16r^2\sin^2\theta$

$z = \pm\frac{1}{5}\sqrt{400 + 400r^2\cos^2\theta + 16r^2\sin^2\theta}$ cylindrical

10. Write an equation of the cylinder $z = 2\sqrt{x^2 + y^2}$ in parametric (surface) form. Use technology to graph the original function, and the parametric form to verify your conversion. (6 points)

$z^2 = 4(x^2 + y^2) = 4r^2$

$z = 2r$

$\vec{r}(u,v) = v\cos u\,\hat{\imath} + v\sin u\,\hat{\jmath} + 2v\,\hat{k}$

11. State the domain and range of the function $f(x,y) = \sqrt{1 + x^2 - 4y^2}$ in appropriate notation. (6 points)

$1 + x^2 - 4y^2 \geq 0$

$x^2 - 4y^2 \leq 1$

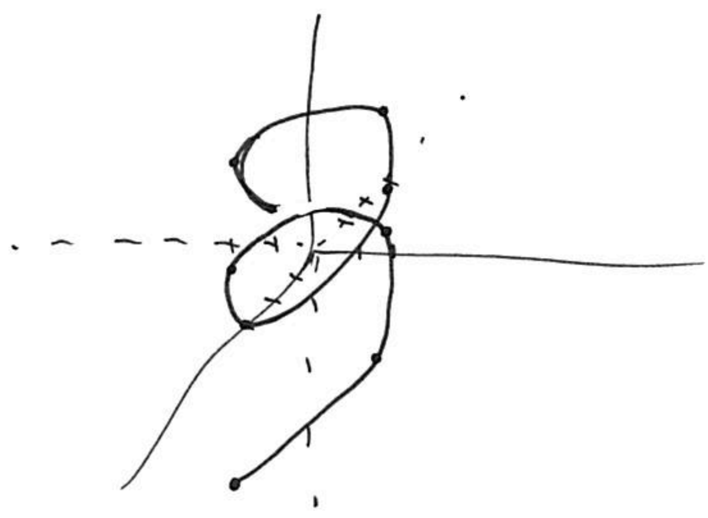
D: $\{(x,y) \mid x^2 - 4y^2 \leq 1\}$

R: $[0, \infty)$

$z \geq 0$

12. Sketch the graph of $\vec{r}(t) = 3\cos t\,\hat{\imath} + 2\sin t\,\hat{\jmath} + \frac{t}{3}\hat{k}$ for two cycles, using a minimum of 10 points. (10 points)

t	x	y	z
-2π	3	0	$-2\pi/3$
$-3\pi/2$	0	2	$-\pi/2$
$-\pi$	-3	0	$-\pi/3$
$-\pi/2$	0	-2	$-\pi/6$
0	3	0	0
$\pi/2$	0	2	$\pi/6$
π	-3	0	$\pi/3$
$3\pi/2$	0	-2	$\pi/2$
2π	3	0	$2\pi/3$



13. Use technology to produce a graph of the vector field $\vec{F}(x, y) = \langle y, y - x \rangle$. Include the graph with your answer, and use that graph to explain the vector displayed at the point (3,1). (8 points)

$\langle 1, 1-3 \rangle = \langle 1, -2 \rangle$

The vector at $(3,1)$ is $\langle 1, -2 \rangle$

Scaled so vectors don't over lap.

Slope of -2

14. Find f_{xyz} of $f(x, y, z) = x^2y^3 + z\sin(x + y) + \sqrt{y^2 - z^2}$. (9 points)

$f_x = 2xy^3 + z\cos(x+y)$

$f_{xy} = 6xy^2 - z\sin(x+y)$

$f_{xyz} = -\sin(x+y)$

15. Consider $\vec{u}(t) = \tan t\,\hat{\imath} + \ln(t)\,\hat{\jmath} + \arcsin 2t\,\hat{k}$. Find: $\vec{u}'(t)$. Describe the continuity of $\vec{u}(t)$. (10 points)

$t \neq$ odd multiples of $\pi/2$ $\quad t \in (0, \infty)$ $\quad t \in [-\frac{1}{2}, \frac{1}{2}]$

D: $(0, 1/2)$

Continuous on that interval

16. Consider $\vec{u}(t) = (1 + 2\sqrt{t})\hat{\imath} + \frac{2t}{1-t^2}\hat{\jmath} + \sec^2 t\,\hat{k}$. Find: $\int \vec{u}(t)\,dt$. (5 points)

$$\int (1+2t^{1/2})\,dt = t + 2 \cdot t^{3/2} \cdot \frac{2}{3} + C_1 = t + \frac{1}{3}t^{3/2} + C_1$$

$$\int \frac{2t}{1-t^2}\,dt \qquad u = 1-t^2 \qquad du = -2t\,dt \qquad \int -\frac{1}{u}\,du \rightarrow -\ln u + C_2$$

$$-\ln|1-t^2| + C_2$$

$$\int \sec^2 t\,dt = \tan t + C_3$$

$$\int \vec{u}(t)\,dt = \left(t + \frac{1}{3}t^{3/2} + C_1\right)\hat{\imath} - (\ln u + C_2)\hat{\jmath} + (\tan t + C_3)\hat{k}$$