MTH 265, Exam #2, Summer 2024 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the volume bounded by the coordinate axes and the plane $3x + 6y + z = 12$. (12 points)

$$\int_0^4 \int_0^{2-\frac{1}{2}x} 12 - 3x - 6y \, dy \, dx =$$

$$\int_0^4 12y - 3xy - 3y^2 \Big|_0^{2-\frac{1}{2}x} =$$

$$\int_0^4 \frac{3}{4}x^2 - 6x + 12 \, dx =$$

$$\frac{1}{4}x^3 - 3x^2 + 12x \Big|_0^4 = 16$$

$z = 12 - 3x - 6y$

$\frac{12 = 3x - 6y}{6}$

$\frac{1}{2}x + y = 2$

$y = 2 - \frac{1}{2}x$

$3x = 12 \Rightarrow x = 4$

2. Set up a double integral to find the area inside the circle $x^2 + y^2 = 9$ and outside $x^2 + y^2 = 6x$. Sketch the graph. Evaluate your integral. (8 points)

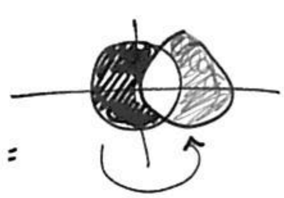
$r = 3$ $\qquad$ $r = 3\cos\theta$

$$A = 2\int_0^{\pi/3} \int_3^{6\cos\theta} r \, dr \, d\theta = \int_0^{\pi/3} r^2 \Big|_3^{6\cos\theta} d\theta =$$

$$\int_0^{\pi/3} 36\cos^2\theta - 9 \, d\theta = \int_0^{\pi/3} 18 + 18\cos 2\theta - 9 \, d\theta =$$

$$\int_0^{\pi/3} 9 + 18\cos 2\theta \, d\theta = 9\theta + 9\sin 2\theta \Big|_0^{\pi/3} =$$

$$9\left(\frac{\pi}{3}\right) + 9\left(\frac{\sqrt{3}}{2}\right) - 0 - 0 = 3\pi + \frac{9\sqrt{3}}{2}$$



2 regions have equal area
right side easier integral

3. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = x^2y\hat{\imath} + xy^2\hat{\jmath} + z^2\hat{k}$ on the path $C: \vec{r}(t) = \frac{1}{3}\sin t\,\hat{\imath} + \frac{1}{2}\cos t\,\hat{\jmath} + t^2\hat{k}$ on $[0,\pi]$. (10 points)

$r'(t) = \frac{1}{3}\cos t\,\hat{\imath} - \frac{1}{2}\sin t\,\hat{\jmath} + 2t\,\hat{k}$

$F(t) = \frac{1}{6}\sin^2 t\cos t\,\hat{\imath} + \frac{1}{18}\sin t\cos^2 t\,\hat{\jmath} + t^4\hat{k}$

$\vec{F}\cdot\vec{dr} = \frac{1}{18}\sin^2 t\cos^2 t - \frac{1}{36}\sin^2 t\cos^2 t + 2t^5 = \frac{1}{36}\sin^2 t\cos^2 t + 2t^5$

$\frac{1}{36}\cdot\frac{1}{2}(1-\cos 2t)\frac{1}{2}(1+\cos 2t) + 2t^5 = \frac{1}{144}(1-\cos^2 2t) + 2t^5 = \frac{1}{144}\cdot\left(1 - \frac{1}{2}(1+\cos 4t)\right)$

$1 - \frac{1}{2} - \frac{1}{2}\cos 4t + 2t^5$

$\frac{1}{2}$

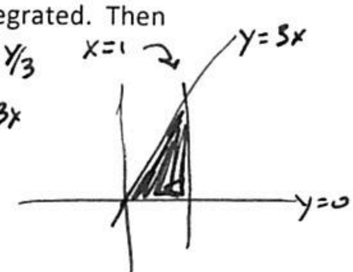
$= \frac{1}{288}(1-\cos 4t) + 2t^5$

$$\int_0^\pi \frac{1}{288} - \frac{1}{288}\cos 4t + 2t^5\,dt = \frac{1}{288}t - \cancelto{0}{\frac{1}{1152}\sin 4t} + \frac{2}{6}t^6\Big|_0^\pi = \boxed{\frac{\pi}{288} + \frac{1}{3}\pi^6}$$

4. Change the order of integration in $\int_0^3\int_{y/3}^1 \frac{y^2}{1+x^4}\,dx\,dy$ so that it can be integrated. Then complete the integration. (10 points)

$X = Y/3$

$Y = 3x$



$$\int_0^1\int_0^{3x} \frac{y^2}{1+x^4}\,dy\,dx = \int_0^1 \frac{1}{3}y^3\cdot\frac{1}{1+x^4}\Big|_0^{3x}dx$$

$$= \int_0^1 \frac{9x^3}{1+x^4}\,dx$$

$u = 1+x^4$

$du = 4x^3dx$

$\frac{1}{4}du = x^3dx$

$$\int \frac{9}{4}\,\frac{1}{u}\,du \Rightarrow \frac{9}{4}\ln|1+x^4|\Big|_0^1 =$$

$$\frac{9}{4}\left[\ln 2 - \cancelto{0}{\ln 1}\right] = \frac{9\ln 2}{4}$$

5. Evaluate the line integral $\int_C 2xyz\,ds$ on the path $\vec{r}(t) = t^2\hat{\imath} + 2t\hat{\jmath} + \sqrt{t}\hat{k}$ on $[0,4]$. (8 points)

$r'(t) = 2t\,\hat{\imath} + 2\hat{\jmath} + \frac{1}{2\sqrt{t}}\hat{k}$

$\|r'(t)\| = \sqrt{4t^2+4+\frac{1}{4t}}\,dt$

$= \sqrt{\frac{16t^3+16t+1}{4t}} = \frac{\sqrt{16t^3+16t+1}}{2t^{1/2}}dt$

$$\int_0^4 2(t^2)(2t)\,t^{1/2}\,ds =$$

$$\int_0^4 \cancelto{2}{4}t^3\cdot\cancel{t^{1/2}}\cdot\frac{\sqrt{16t^3+16t+1}}{\cancel{2t^{1/2}}}\,dt =$$

$$\int_0^4 2t^3\sqrt{16t^3+16t+1}\,dt$$ integrate numerically

$$\approx 3123.592654\ldots$$

6. Determine if the vector field $\vec{F}(x,y,z) = yz\cos x\,\hat{\imath} + (2y + z\sin x)\hat{\jmath} + (y\sin x - 1)\hat{k}$ is conservative. (5 points)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz\cos x & 2y+z\sin x & y\sin x - 1 \end{vmatrix} = (\sin x - \sin x)\,\hat{i} - (y\cos x - y\cos x)\,\hat{j} + (z\cos x - z\cos x)\,\hat{k} = \vec{0}$$

it is conservative

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

7. Use technology to graph each of the vector field equations. Attach the graphs. On each graph note any curves where one component is zero (horizontal or vertical vectors: these are called nullclines). Determine if either of the fields is conservative. (20 points)

   a. $\vec{F}(x,y) = \sin x\,\hat{\imath} + \cos y\,\hat{\jmath}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin x & \cos y & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (0-0)\hat{k} = \vec{0}$$

Conservative

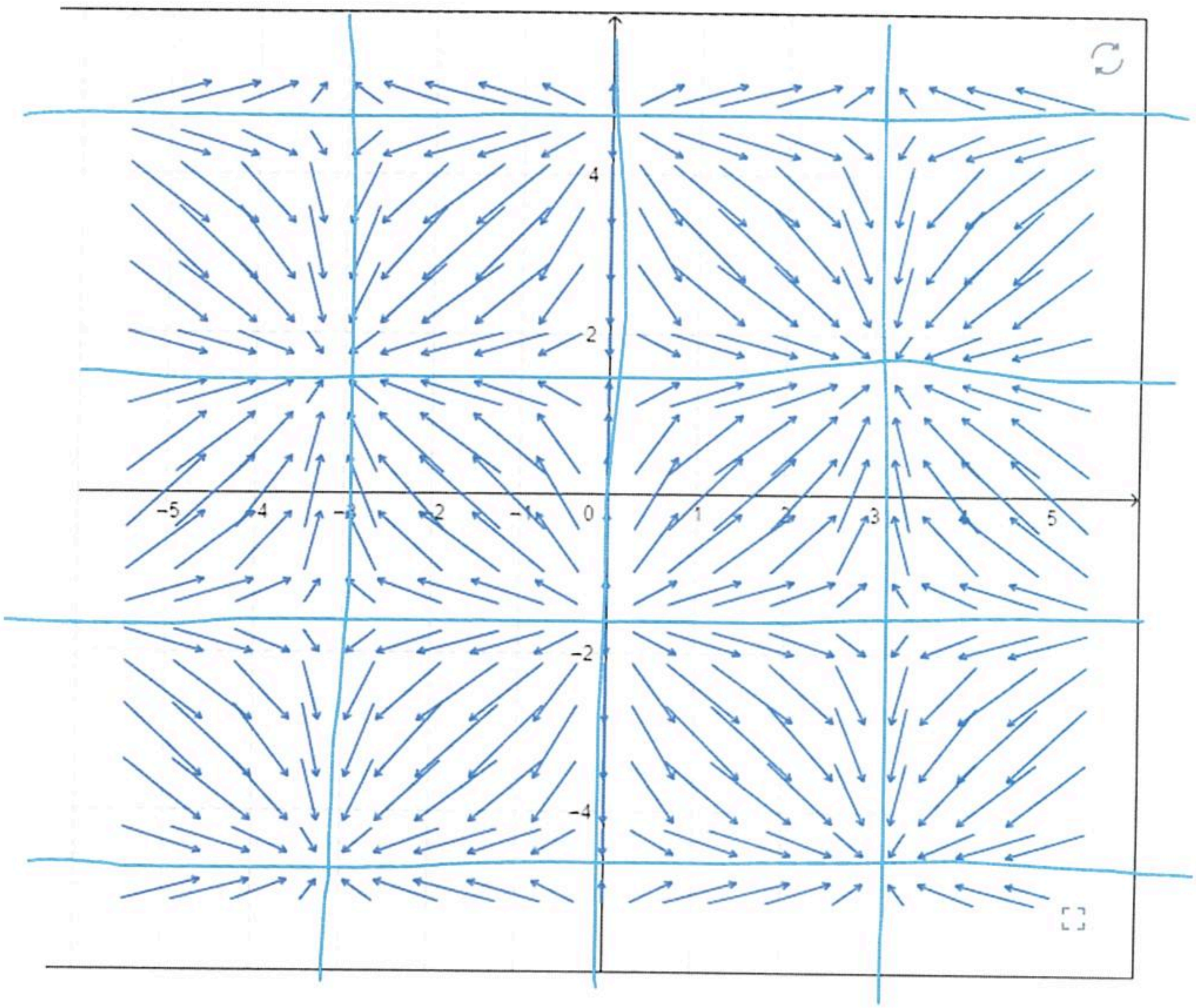
graph attached

   b. $\vec{F}(x,y) = 3x\hat{\imath} - 2y\hat{\jmath}$

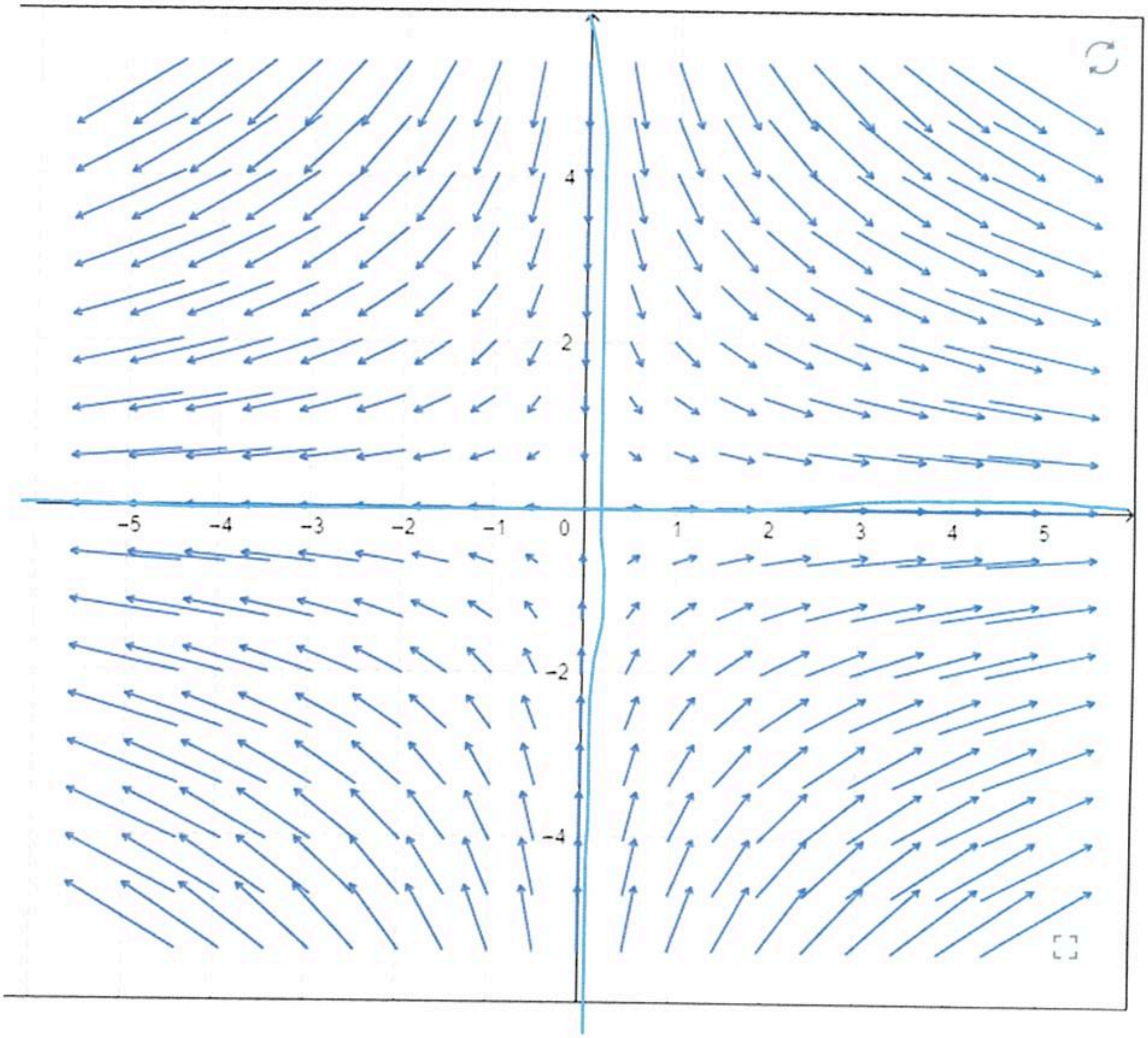
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x & -2y & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (0-0)\hat{k} = \vec{0}$$

Conservative

graph attached



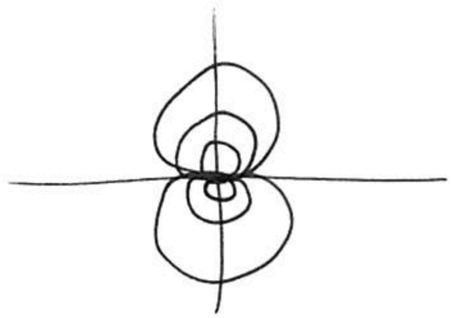
4
2
−2
−4
−5
−4
−3
−2
−1
0
1
2
3
4
5

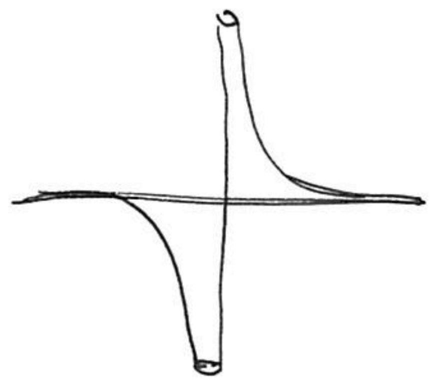


−5
−4
−3
−2
−1
0
1
2
3
4
5
4
2
−2
−4

8. Sketch the level curves for $f(x,y) = \frac{4y}{x^2+y^2}$ for values of z in $[-2,2]$. Use technology to create a graph of the curve, and to verify your level curves. Explain in your own words how the two graphs are related. (10 points)

$$k = \frac{4y}{x^2+y^2} \rightarrow x^2+y^2 = \frac{4y}{k}$$





Circles get bigger as k gets smaller

these are crosssectional slices at different values of z

graph approaches the plane as z → 0 undefined as the radius goes to zero.

9. Convert the triple integrals to the given coordinate systems and then complete the integration. Describe the region being integrated (over). (10 points each)

   a. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2-1}^{\overset{1-r^2}{1-x^2-y^2}} xdzdydx$ in cylindrical.

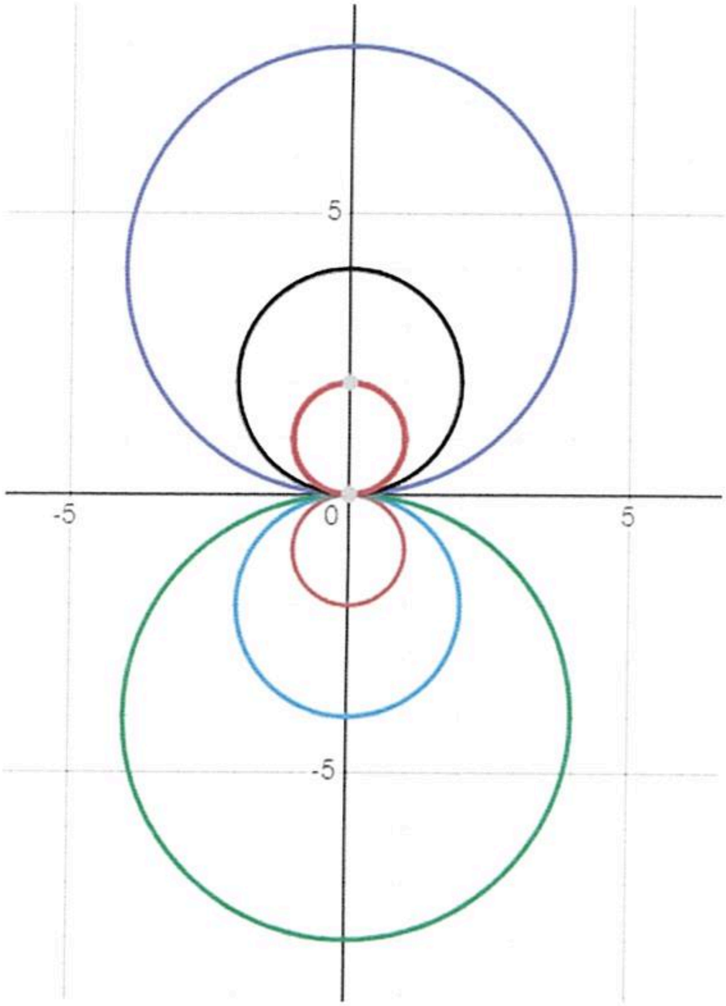
$r=1$ $\quad r^2-1$

$0 = \theta = 2\pi$

$$\int_0^{2\pi}\int_0^1\int_{r^2-1}^{1-r^2} r\cos\theta\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^2\cos\theta\, dr\, d\theta\, z\Big|_{r^2-1}^{1-r^2}$$

$$\int_0^{2\pi}\int_0^1 r^2\cos\theta\,\underbrace{(1-r^2-r^2+1)}_{2-2r^2}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (2r^2-2r^4)\cos\theta\, dr\, d\theta =$$

$$\int_0^{2\pi} \frac{2}{3}r^3 - \frac{2}{5}r^5\Big|_0^1 \cos\theta\, d\theta = \frac{4}{15}\int_0^{2\pi}\cos\theta\, d\theta = \frac{4}{15}\sin\theta\Big|_0^{2\pi} = 0$$



5
-5
0
5
-5

$\rho = 4$

b. $\int_{-4}^{4}\int_{0}^{\sqrt{16-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \frac{e^{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\, dzdydx$ in spherical.

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^{4} \frac{e^{\rho}}{\cancel{\rho^2}}\cancel{\rho^2}\sin\varphi \, d\rho\, d\varphi\, d\theta = \int_0^{\pi}\int_0^{\pi/2} e^{\rho}\Big|_0^4 \sin\varphi\, d\varphi\, d\theta =$$

$$(e^4-1)\pi \int_0^{\pi/2} \sin\varphi\, d\varphi = (e^4-1)\pi\,(-\cos\varphi\Big|_0^{\pi/2} =$$

$$(e^4-1)\pi(-0+1) = (e^4-1)\pi$$

10. Set up a double integral to find the volume under $f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ on the region $x^2+y^2 \le 9, x \ge 0$. Do not integrate. (8 points)

$$\int_{-\pi/2}^{\pi/2}\int_0^3 \frac{\ln r^2}{r}\, r\, dr\, d\theta =$$

$$\int_{-\pi/2}^{\pi/2}\int_0^3 2\ln r \, dr\, d\theta$$

11. Given the vector field $\vec{F}(x,y,z) = yz\cos x\,\hat{\imath} + (2y + z\sin x)\hat{\jmath} + (y\sin x - 1)\hat{k}$, find the potential function if it exists; if not, explain why not. (5 points)

$$\int yz\cos x\, dx = yz\sin x + f(y,z)$$

$$\int 2y + z\sin x\, dy = y^2 + yz\sin x + g(x,z)$$

$$\int y\sin x - 1\, dz = yz\sin x - z + h(x,y)$$

$$\varphi = yz\sin x + y^2 - z + K$$

12. Find ∇f and $\nabla^2 f$ for $f(x,y,z) = 2x^3 z \sin y + \sqrt[4]{z}$. (8 points)

$$\nabla f = \left\langle 6x^2 z \sin y,\ 2x^3 z \cos y,\ 2x^3 \sin y + \frac{1}{4\sqrt[4]{z^3}} \right\rangle$$

$$\frac{1}{4} z^{-3/4}$$

$$\nabla f^2 = 12xz \sin y - 2x^3 z \sin y - \frac{3}{16} z^{-7/4}$$

13. Find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ for $\vec{F}(x,y,z) = \cos(xy)\hat{\imath} + \sec(yz)\,\hat{\jmath} + \frac{x}{z}\hat{k}$. (10 points)

$$\vec{\nabla} \cdot \vec{F} = -y \sin(xy) + z \sec yz \tan yz - \frac{x}{z^2}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos xy & \sec yz & \frac{x}{z} \end{vmatrix} = (0 - y \sec yz \tan yz)\hat{\imath} - \left(\frac{1}{z} - 0\right)\hat{\jmath} + (0 - 0)\hat{k}$$

$$-y \sec yz \tan yz\, \hat{\imath} - \frac{1}{z}\hat{\jmath} + 0\hat{k}$$

Cylindrical

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$

Spherical

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \theta$$
$$\cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \phi$$
$$x^2 + y^2 = \rho^2\sin^2\phi = r^2$$

Dels

$$\frac{\partial}{\partial x} = \text{ partial derivative with respect to } x$$
$$\nabla f = grad\ f$$
$$\nabla^2 f = \nabla\cdot(\nabla f) = Laplacian\ of\ f$$
$$\nabla\cdot\vec{F} = div\ \vec{F}$$
$$\nabla\times\vec{F} = curl\ \vec{F}$$

Misc

$$ds = \|\vec{r}'(t)\|dt$$