5/22/2024

Lines and Planes in Space
Quadric Surfaces
Functions of Several Variables
Cylindrical and Spherical Coordinates
Parametric Surfaces
Limits in 2+ Variables

Tetrahedron: like a parallelepiped, it can be defined by three vectors coming out of one vertex. And so like the parallelepiped, we can find the volume using the magnitude of the triple scalar product, but

$$V_{tetrahedron} = \frac{1}{6} V_{parallelepiped}$$

Lines in Space
Typically, these must be represented in parametric form. Another way to represent a line or curve in space is as the intersection of two or more surfaces.

Parametric form:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

Vector valued function form:

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Find the parametric equation of the line passing through the points (1,3,2), and (-2,1,5)

$$\vec{u} = \langle \Delta x, \Delta y, \Delta z \rangle = \langle a, b, c \rangle$$

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}$$

$$\vec{u} = \langle -3, -2, 3 \rangle$$
$$\vec{r}(t) = \langle -3t + 1, -2t + 3, 3t + 2 \rangle$$

When t=0, you are at the first point, and when t=1, you are at the second point.

Symmetric form of the line:
Derived from solving the parametric forms for t, and then setting all the t's equal to each other.

$$\frac{(x - x_0)}{a} = \frac{(y - y_0)}{b} = \frac{z - z_0}{c}$$

If the rate of change for one of the variables is 0, you can't write the symmetric equation because you divide by 0.

Suppose I have a line with the direction vector $\langle 1, 3, 0 \rangle$, passing through the point (4,-1,6).

Parametric/vector-valued function form does not change:

$$x = 4 + t, y = 3t - 1, z = 6$$

In “symmetric form”, you can’t divide by zero:

$$\frac{x-4}{1} = \frac{y+1}{3}, z = 6$$

Planes: the simplest surface that you can have in 3 variables (in 3D) is a plane.
Any linear equation in 3D is a plane.
Is defined by vector perpendicular to the plane, $\vec{n}$, and any point that the plane passes through.

Find the equation of the plane passing through the point (4,-3,1) and which is normal to the vector $\langle 2,8,-11\rangle$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$2(x - 4) + 8(y + 3) - 11(z - 1) = 0$$

Alternative versions of the plane equation question:

1) Give you three points in the plane
    a. Take the points in pairs and make vectors that lie in the plane
    b. Take the cross product of the two vectors and get the perpendicular vector
    c. Put that vector and one of the original points into the plane equation
2) Give you vectors in the plane, and a point
    a. Take the cross product to get the perpendicular vector
    b. Put that and one point into the equation
3) One line and a point not on the line
    a. Find a point on the line.
    b. Make a new vector in the plane from the extra point to the one on the line
    c. Now with two vectors, continue version 2
4) Another plane that has some relationship to the new plane
5) A line perpendicular to the plane

If you have an equation of the plane like $Ax + By + Cz = D$, then the normal vector to the plane is just $\langle A, B, C\rangle$.

Distance between a point and a line
P is a point in space
Q is a point on the line
$\vec{u}$ is the direction vector for the line.
Distance between the point and the line:

$$D = \frac{\|\overrightarrow{PQ} \times \vec{u}\|}{\|\vec{u}\|}$$

Suppose I have the line: $\vec{r}(t) = \langle 3t + 1, 2t - 2, t + 4\rangle$
And I want to find the shortest distance to the point (3,1,1)=P

Q=(1,-2,4)

$$\overrightarrow{PQ} = \langle -2, -3, 3\rangle$$
$$\vec{u} = \langle 3, 2, 1\rangle$$
$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -3 & 3 \\ 3 & 2 & 1 \end{vmatrix} = (-3-6)\hat{\imath} - (-2-9)\hat{\jmath} + (-4+9)\hat{k} = \langle -9, 11, 5\rangle$$

$$D = \frac{\sqrt{81+121+25}}{\sqrt{9+4+1}} = \frac{\sqrt{227}}{\sqrt{14}}$$

Distance between a point and a plane
P is a point in space
Q is a point on the plane
$\vec{n}$ is the normal vector to the plane
Distance between the point and the plane:

$$D = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

Find the distance between the point (1,6,-3) and the plane $3x - 2y - 4z = 12$

$$\vec{n} = \langle 3, -2, -4\rangle$$

Q=(4,0,0)

$$\overrightarrow{PQ} = \langle 3, -6, 3\rangle$$

$$D = \frac{9+12-12}{\sqrt{9+4+16}} = \frac{9}{\sqrt{29}}$$

Last thing to note about planes:
To find the angle of intersection between two planes, find the normal vectors for both planes, and then use the dot product formula for cosine:

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\|\vec{n}_2\|}$$

Quadric Surfaces:
Rotated conics (in 3D)

Sphere:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = \rho^2$$

Ellipsoid:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1$$

Paraboloid: one component is linear, that linear component determines the orientation

Hyperbolic paraboloid and a elliptic paraboloid

Elliptic paraboloid:

$$\frac{(x-h)^2}{a^2}+\frac{(y-h)^2}{b^2}=\frac{z}{c}$$

This version opens up when c is positive, and down when c is negative.

If the linear component is y, then the graph opens in the y-orientation (in the direction of the y-axis

Hyperbolic paraboloid is a hyperbola on the left side and linear on the other.

$$\frac{(x-h)^2}{a^2}-\frac{(y-h)^2}{b^2}=\frac{z}{c}$$

This is the classic saddle graph.

Hyperboloid: Of one sheet, and the hyperboloid of two sheets
Hyperboloid of one sheet looks like a nuclear power plant cooling tower
Hyperboloid of two sheets, one side is the shape of a hyperbolic microphone

One sheet has only one negative
Two sheets has two negatives.

One sheet:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=1$$

Two sheets:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=1$$

One sheet, the negative sign is indicating the axis the graph is rotated around. In our example, that's z. In the two sheet case, the positive variable, is the orientation of the transverse axis. In this example, the x-axis passes through both sheets.

Cone:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=0$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\frac{(z-l)^2}{c^2}$$

Cylinders:
Essentially any 2D equation in 3 dimensions.

$$x^2+y^2=4$$

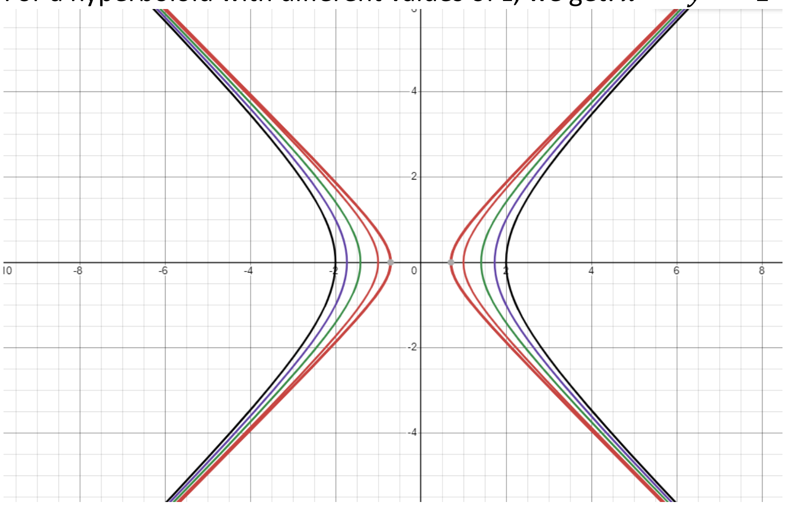
$$y=x^2$$

A way to plot three-dimensional functions in two dimensions is called level curves (contour curves): choose a value for z, and then plot the resulting two-variable equation.

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 = c^2$$

For a hyperboloid with different values of z, we get: $x^2 - y^2 = z^2$



Functions of several variables:
Function in several variables means that one variable (the function variable) can be expressed as a function of the remaining variables which are independent.

$$z = f(x, y)$$
$$w = f(x, y, z)$$

$$f(x, y) = x^2 + 2y - 3$$

$$f(1,3) = 1^2 + 2(3) - 3 = 1 + 6 - 3 = 4$$

Domain and range of multivariable functions.
The range won't change that much from the one-variable case
The domain however will depend on two or more variables so the notation does have to change

$$Domain: \{(x, y)|conditions\}$$

$$f(x, y) = \sqrt{2x - 3y + 11}$$

State the domain and range of f.

$$Domain: \{(x, y)|2x - 3y + 11 \geq 0\}$$

Range:

$$[0, \infty)$$

$$f(x,y) = \sqrt{x} - \ln y$$

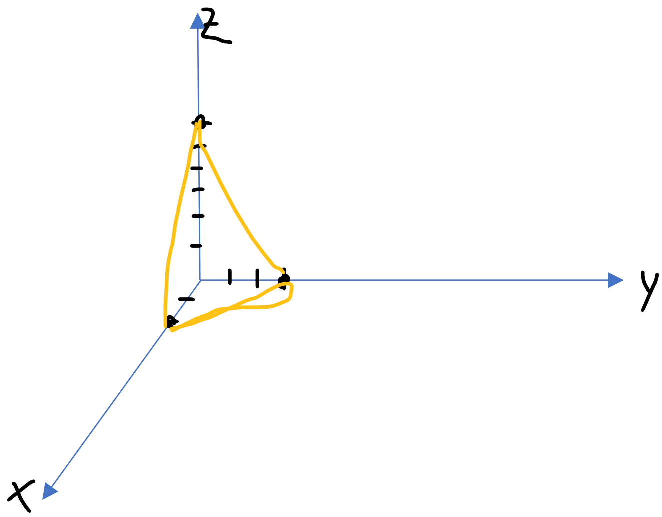
$$Domain: \{(x,y)|x \geq 0, y > 0\}$$

Range: $(-\infty, \infty)$

Plotting a plane in 3D:
If the plane cuts through the first octant, then we can sketch that part of the plane

$$f(x,y) = 6 - 3x - 2y$$
$$3x + 2y + z = 6$$

All the intercepts are plotable on the standard axis.



x-intercept has both y=0, and z=0
y=intercept has both x=0, z=0
z=intercept has both x=0, y=0

Switching to Cylindrical and Spherical Coordinates

Cylindrical is basically just polar +z
In the plane, that's a polar graph, and the z-coordinate is unchanged.

$$x = r\cos\theta\,, y = r\sin\theta\,, x^2 + y^2 = r^2, \theta = \tan^{-1}\left(\frac{y}{x}\right), z = z$$

Spherical

$$x = \rho\cos\theta\sin\phi\,, y = \rho\sin\theta\sin\phi\,, z = \rho\cos\phi$$
$$x^2 + y^2 + z^2 = \rho^2, \theta = \tan^{-1}\left(\frac{y}{x}\right), \phi = \cos^{-1}\left(\frac{z}{x^2 + y^2 + z^2}\right)$$
$$x^2 + y^2 = r^2 = \rho^2\sin\phi$$

ρ is the function variable, ϕ is the angle to the point from the positive z-axis (between 0 and π on the negative z-axis)

$$z = 4x^2 + 4y^2 - 6$$

Convert to cylindrical, then to spherical

$$z = 4(x^2 + y^2) - 6 = 4r^2 - 6$$

To spherical

$$z = 4(x^2 + y^2) - 6$$
$$\rho \cos\phi = 4\rho^2 \sin\phi - 6$$

Cones:

$$x^2 + y^2 = z^2$$

To cylindrical

$$r^2 = z^2$$
$$r = z$$

In spherical:

$$\rho^2 \sin^2\phi = \rho^2 \cos^2\phi$$

$$\tan^2\phi = 1$$
$$\tan\phi = 1$$
$$\phi = \frac{\pi}{4}$$

In cylindrical (r, θ, z)
In spherical (ρ, θ, ϕ)

Parametric surfaces:
Vector-valued functions, but of two variables rather than one

$$\vec{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}$$

Explicit functions, swap x with u, and y with v and then z=f(u,v)

$$f(x, y) = 2x^2 - 3y^2 + 1$$

$$\vec{r}(u, v) = \langle u, v, 2u^2 - 3v^2 + 1\rangle$$

What if I have a cone?

$$x^2 + y^2 = z^2$$

To cylindrical:

$$r^2 = z^2$$
$$r = z$$
$$x = r\cos\theta\,, y = r\sin\theta$$

$$\vec{r}(u, v) = \langle u\cos v\,, u\sin v\,, u\rangle$$

Sphere:

$$x^2 + y^2 + z^2 = 9$$

$$x = \rho \cos\theta \sin\phi \, , y = \rho \sin\theta \sin\phi \, , z = \rho \cos\phi$$
$$\vec{r}(u, v) = \langle 3\cos u \sin v \, , 3 \sin u \sin v \, , 3\cos v \rangle$$

Limits in more than one variable

If you are at the point on a function that is well-defined at that point, just plug in the variable.
If possible, try to make a substitution to reduce the number of variables in the problem, particularly for points that are not defined. L'Hopital's can only be used on one variable functions, so if you can make a substitution to reduce the number of variables, then you can use that technique.

You can also use other coordinate systems like polar (or spherical) to reduce the problem.

We will continue with limits next class.