5/28/2024

Line Integrals
Partial Derivatives
Total Differential

Homework #3, problem #5 has some conversion errors in the formulas that may not be displaying correctly. Here are the problems again. Use the directions given in the homework (those are displaying okay).

5a. $\vec{r}(t) = t\hat{\imath} + (2t - 4)\hat{\jmath} + (3t - 7)\hat{k}$
5b. $\vec{r}(t) = \sin(t)\hat{\imath} + \cos(t)\hat{\jmath} + t\hat{k}$ (helix)
5c. $\vec{r}(t) = t^2\hat{\imath} + t\hat{\jmath} + 4\hat{k}$
5d. $\vec{r}(t) = \sin^2(t)\,\hat{\imath} + t\hat{\jmath} + \cos(t^2)\,\hat{k}$

Vector-Valued Functions
Review:

$$\vec{r}(t) = \sin^2(t)\,\hat{\imath} + t\hat{\jmath} + \cos(t^2)\,\hat{k}$$
$$\vec{r}\,'(t) = \langle 2\sin(t)\cos(t), 1, -2t\sin(t^2)\rangle$$

$$\vec{r}(t) = t\hat{\imath} + (2t - 4)\hat{\jmath} + (3t - 7)\hat{k}$$
$$\int \vec{r}(t)\,dt = \langle \frac{t^2}{2} + C_1, t^2 - 4t + C_2, \frac{3}{2}t^2 - 7t + C_3\rangle$$

How do we integrate over curves in space? How do we integrate over a path through a vector field?

Line Integral (16.2/6.2)

$$\int_C f(x,y)ds\,, \int_C f(x,y,z)ds$$

$$s = \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; or \; s = \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\int_C ds = \int_C \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2}\;dt$$
$$\int_C ds = \int_C \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\;dt$$

Convert the $f(x,y)$ or $f(x,y,z)$ into a function of t alone by replacing the path components for x, y, and z.

This step is what is going to give us a one-variable integra.

$C: \vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$, find the value of the line integral $\int_C (2x^2 + 2y^2)ds$ along the path between $0 \le t \le 2\pi$

$$ds = \|\vec{r}'(t)\|dt = \|\langle -\sin(t), \cos(t), 1\rangle\|dt = \sqrt{\sin^2 t + \cos^2 t + 1}dt = \sqrt{2}dt$$

$$\int_C (2x^2 + 2y^2)ds = \int_0^{2\pi} 2(\cos^2 t + \sin^2 t)\sqrt{2}dt = \int_0^{2\pi} 2\sqrt{2}dt = 4\pi\sqrt{2}$$

The mass density of the wire varies by the function $f(x, y, z) = (2x^2 + 2y^2)$, and we want to find the total mass of the wire following the curve C, on the given interval.

$$\int_C P(x, y)dx + Q(x, y)dy \ \ or \ \int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

Differential notation version

Our path is still expressed in terms of the parameter t, and we still need to replace the functions of x and y (and z) with expressions in t taken from the path. $dx = x'(t)dt, dy = y'(t)dt, dz = z'(t)dt$ where $x(t), y(t), z(t)$ are taken from the path.

Evaluate $\int_C y^2dx + xdy$ where C is the line segment from (-5,-3) to (0,2).

$$\vec{r}(t) = \langle at + x_0, bt + y_0\rangle$$

$$a = 0 - (-5) = 5$$
$$b = 2 - (-3) = 5$$
$$\vec{r}(t) = \langle 5t - 5, 5t - 3\rangle, 0 \le t \le 1$$
$$\vec{r}'(t) = \langle x'(t), y'(t)\rangle = \langle 5,5\rangle$$

$$\int_C y^2dx + xdy = \int_0^1 (5t - 3)^2(5)dt + (5t - 5)(5)dt = \int_0^1 5[(5t - 3)^2 + (5t - 5)]dt =$$

$$5\int_0^1 25t^2 - 30t + 9 + 5t - 5 \, dt = 5\int_0^1 25t^2 - 25t + 4dt = 5\left[\frac{25}{3}t^3 - \frac{25}{2}t^2 + 4t\right]_0^1 =$$

$$5\left[\frac{25}{3} - \frac{25}{2} + 4\right] = -\frac{5}{6}$$

What if we tried a different path:

Evaluate $\int_C y^2dx + xdy$ where C is the parabolic curve $x = 4 - y^2$ from (-5,-3) to (0,2).

$$\vec{r}(t) = \langle 4 - t^2, t\rangle \ on - 3 \le t \le 2$$
$$\vec{r}'(t) = \langle -2t, 1\rangle$$

$$\int_C y^2dx + xdy = \int_{-3}^{2} (t^2)(-2tdt) + (4 - t^2)(1dt) = \int_{-3}^{2} -2t^3 - t^2 + 4dt =$$

$$-\frac{1}{2}t^4 - \frac{1}{3}t^3 + 4t\Big|_{-3}^{2} = \left[-\frac{1}{2}(16) - \frac{1}{3}(8) + 8\right] - \left[-\frac{1}{2}(81) - \left(\frac{1}{3}\right)(-27) + 4(-3)\right] =$$

$$\left[-8 - \frac{8}{3} + 8\right] - \left[-\frac{81}{2} + 9 - 12\right] = \frac{245}{6}$$

In some cases, even starting and stopping in the same place, you may not get the same answer if you travel a different path from point to point.

Third version involves a vector field

$$\int_C \vec{F}(x,y,z) \cdot d\vec{r} = \int_C \vec{F}(x,y,z) \cdot \vec{r}\,'(t)dt = \int_C \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle \cdot \langle dx, dy, dz\rangle =$$
$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle xy, yz, zx\rangle$ and the curve is given by $\vec{r}(t) = \langle t, t^2, t^3\rangle, 0 \le t \le 1$

$$\vec{F}(x,y,z) \to \vec{F}(t) = \langle t(t^2), (t^2)(t^3), t(t^3)\rangle = \langle t^3, t^5, t^4\rangle$$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2\rangle$$

$$\int_0^1 \langle t^3, t^5, t^4\rangle \cdot \langle 1, 2t, 3t^2\rangle dt = \int_0^1 t^3 + 2t^6 + 3t^6 dt = \int_0^1 t^3 + 5t^6 dt =$$

$$\frac{t^4}{4} + \frac{5}{7}t^7\Big|_0^1 = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$$

Partial Derivatives
They take the "derivative" or measure the rate of change in one variable at a time

	Leibniz	"Prime" Notation
Single Variable	$\frac{dy}{dx}, \frac{df}{dx}, \frac{d^2f}{dx^2}$	$f'(x), y', y'', f'''(x)$
Multivariable	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^3 f}{\partial x \partial y^2}$	$f_x(x,y), z_x, f_y(x,y), z_y, f_{yyx}(x)$

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

$$f(x,y) = x^2 - 3xy + 3y^2 + 4y$$

Find $f_x(x,y)$ using the definition of the derivative.

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 3(x + \Delta x)y + 3y^2 + 4y - (x^2 - 3xy + 3y^2 + 4y)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 3xy - 3y\Delta x + 3y^2 + 4y - x^2 + 3xy - 3y^2 - 4y}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 - 3y\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 3y)}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x - 3y = 2x - 3y$$

The result suggests that any term that contains x take the derivative of x normally, and treat y like a constant.

$$f(x,y) = x^2 - 3xy + 3y^2 + 4y$$
$$f_x(x,y) = 2x - 3y$$
$$f_y(x,y) = -3x + 6y + 4$$

$$f_{xx}(x,y) = 2$$
$$f_{yy}(x,y) = 6$$
$$f_{xy}(x,y) = -3$$
$$f_{yx}(x,y) = -3$$

$$f_{xxyy}(x,y) = f_{xyyx}(x,y) = f_{yxyx}(x,y) = f_{yyxx}(x,y)$$

If you have a two-variable function, there two first partial derivatives f_x, f_y
If you have a function of three variables, there are three first partial derivatives, f_x, f_y, f_z

If you have a function of two variables, there are 4 second partial derivatives: $f_{xx}, f_{yy}, f_{xy} = f_{yx}$
If you have a function of three variables, there are 9 second partial derivatives: $f_{xx}, f_{yy}, f_{zz}, f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$

Total differential:
Recall from one-variable differentials:

$$dy = f'(x)dx$$

$$\Delta y \approx f'(x)\Delta x$$

$$y + \Delta y \approx f(x) + f'(x)\Delta x$$

$$dz = f_x(x,y)dx + f_y(x,y)dy$$
$$dw = f_x(x,y,z)dz + f_y(x,y,z)dy + f_z(x,y,z)dz$$

$$\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

Suppose that $f(x,y) = x^2 + 3xy - y^2$

a) Find the differential dz
b) Use the differential to estimate the value of the change from $(2,3)\ to\ (2.05, 2.96)$

$$dz = f_x(x,y)dx + f_y(x,y)dy$$
$$f_x(x,y) = 2x + 3y$$
$$f_y(x,y) = 3x - 2y$$

$$dz = (2x + 3y)dx + (3x - 2y)dy$$

$$\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

$$\Delta x = 0.05, \Delta y = -0.04$$

Plug into the derivatives, the nice point.

$$\Delta z \approx \big(2(2) + 3(3)\big)0.05 + \big(3(2) - 2(3)\big)(-0.04) = (13)(0.05) + (0)(-0.04) = 0.65$$

Estimate for $f(2.05, 2.96) \approx f(2,3) + 0.65 = 4 + 18 - 9 + 0.65 = 13.65$

Compare to the true value of $f(2.05, 2.96) = 13.6449$

The first exam is next Monday, and the material for Exam #1 ends here.