5/29/2024

Gradients, Del Notation, Traces/Level Curves
Conservative Vector Fields, Potential Functions/Fundamental Theorem of Line Integrals

Last time we discussed little del: ∂
Tonight we are looking at capital del: ∇

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

If we apply this operator to a single function, we transform a multivariable function into a vector field.
If we apply this operator to a vector field, we can apply it with the dot product operation to create a multivariable function, or we can apply it with the cross product operation and obtain a new vector field.

Gradient:
Grad(f) = $\nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

Divergence:
Div(F) = $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

Curl:
Curl(F) = $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

Laplacian:

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$$

Example.

$$f(x, y, z) = x^2 + 5xyz - y^2 z^3 - 3z$$

This is a multivariable function, and so we can apply the gradient and the Laplacian to this function.

$$\nabla f = \langle 2x + 5yz, 5xz - 2yz^3, 5xy - 3y^2 z^2 - 3 \rangle$$

$$\nabla^2 f = 2 - 2z^3 - 6y^2 z$$

Example.

$$\vec{F}(x, y, z) = \langle xy^2 z^3, x^3 yz^2, x^2 y^2 z \rangle$$

The divergence applied to a vector field will generate a function, and the curl applied to the vector field creates another vector field.

$$\vec{\nabla}\cdot\vec{F} = \langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\rangle\cdot\langle xy^2z^3, x^3yz^2, x^2y^2z\rangle = \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^2z) =$$

$$y^2z^3 + x^3z^2 + x^2y^2$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^2z \end{vmatrix} =$$

$$\left(\frac{\partial}{\partial y}(x^2y^2z) - \frac{\partial}{\partial z}(x^3yz^2)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(x^2y^2z) - \frac{\partial}{\partial z}(xy^2z^3)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(xy^2z^3)\right)\hat{k}$$

$$= \langle 2x^2yz - 2x^3yz, -2xy^2z + 3xy^2z^2, 3x^2yz^2 - 2xyz^3\rangle$$

Gradient field:
The components of the gradient are the first partial derivatives in each component of the vector. The direction of the gradient represents the direction of greatest change in the function value.

In the gradient field, when all the vectors point toward a certain point, that point is a maximum. In the gradient field, when all the vectors point away from a certain point, that point is a minimum. In the gradient field, when some vector point toward the point, and some point away from the point, that point is a saddle point.

$$f(x,y) = x^2 + y^2 - 4x$$

Find the gradient field:

$$\nabla f = \langle 2x-4, 2y\rangle$$

Plot the places on the field where one component of the vector is 0.

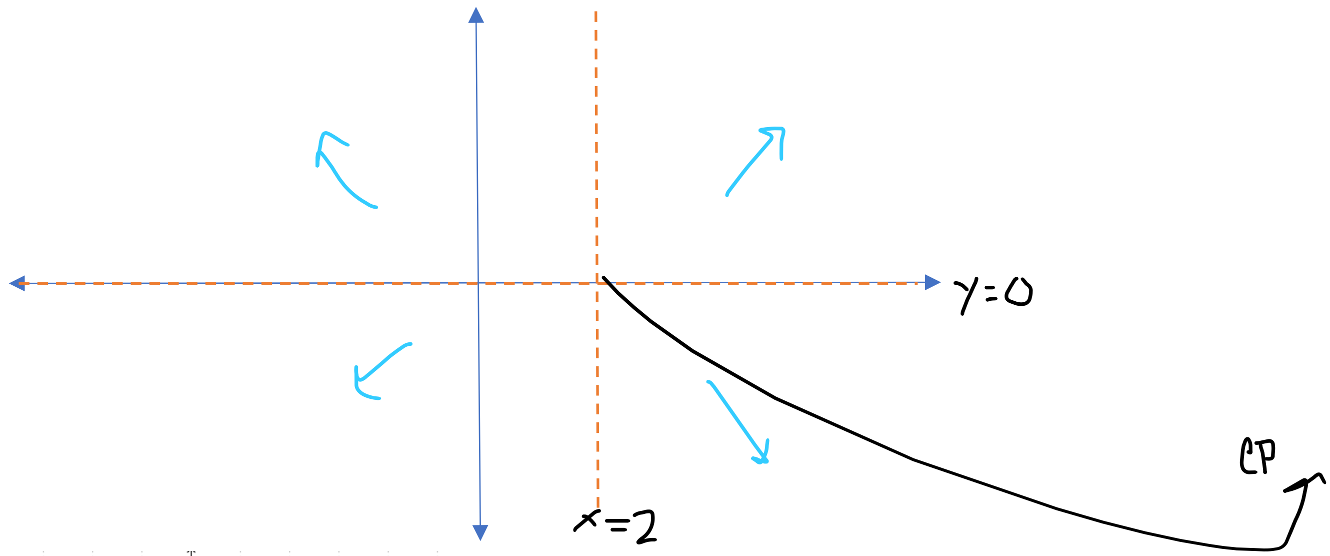
$$2x - 4 = 0$$
$$x = 2$$
$$2y = 0$$
$$y = 0$$

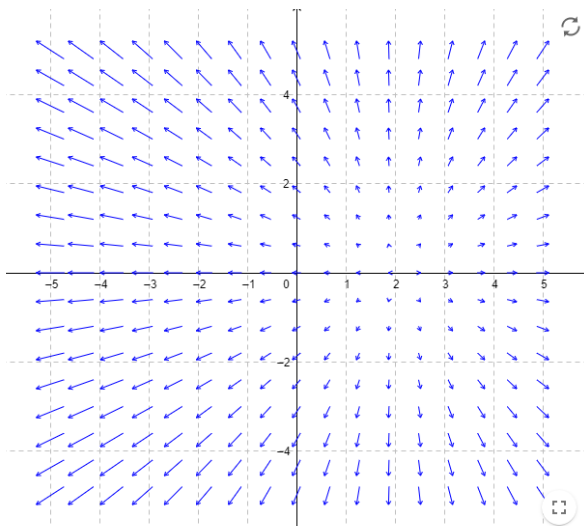
The critical points will occur when both derivatives are 0, where these two lines cross.

The vectors can only change direction (i.e. pointing at top right to pointing at top left, or bottom right, etc.) by crossing over one of our zero-lines. So plot one vector in each of the 4 sections of the graph.

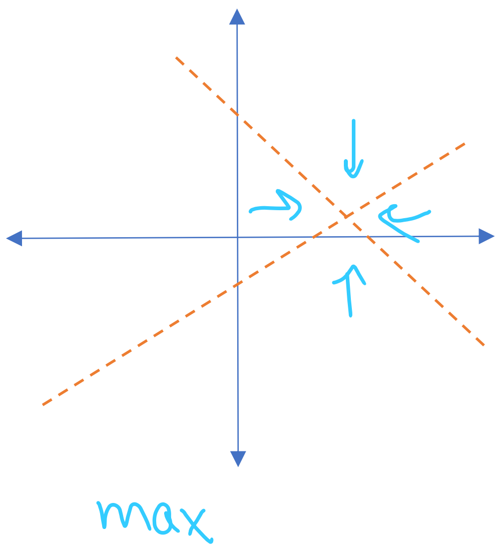
x	y	∇f
3	1	$\langle 2,2\rangle$
0	1	$\langle -4,2\rangle$
0	-1	$\langle -4,-2\rangle$
3	-1	$\langle 2,-2\rangle$

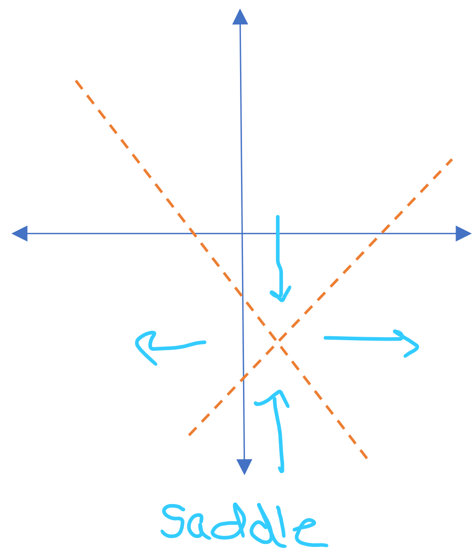
All the arrows are pointing away from the critical point. And since the gradient vector points toward the direction of greatest INCREASE, this critical point is a minimum.





What would a maximum or a saddle point look like?





This is essentially the equivalent of the first derivative test for multivariable functions.

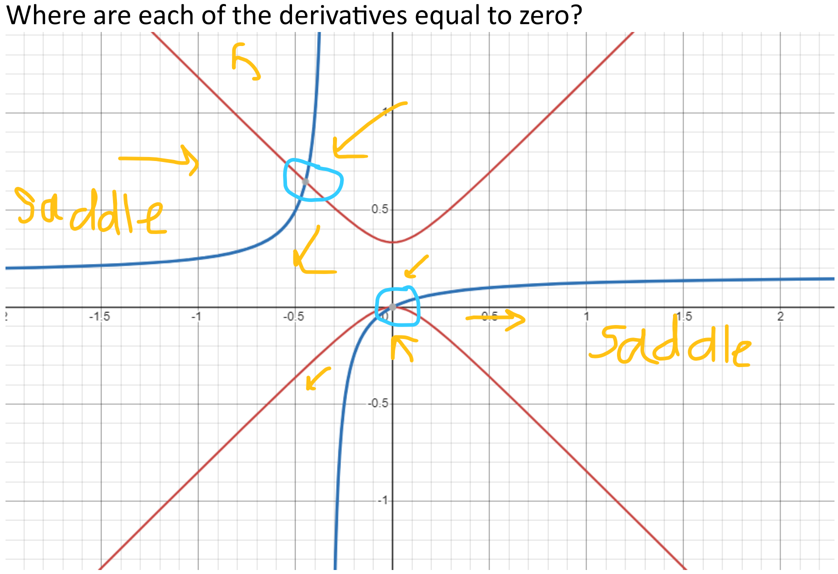
Connection between gradient fields and level curves:

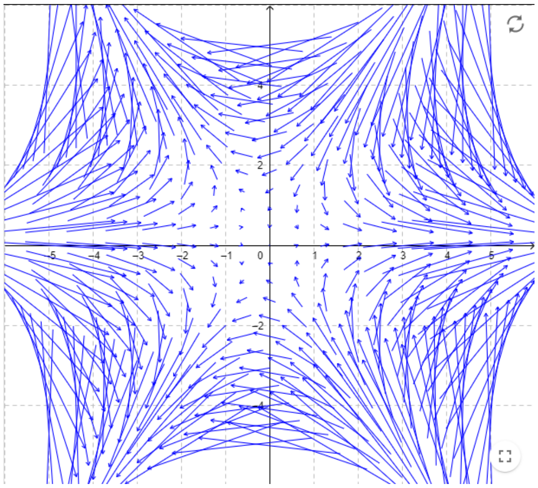
$$f(x,y) = x^3 - 3xy^2 + xy - y^2$$

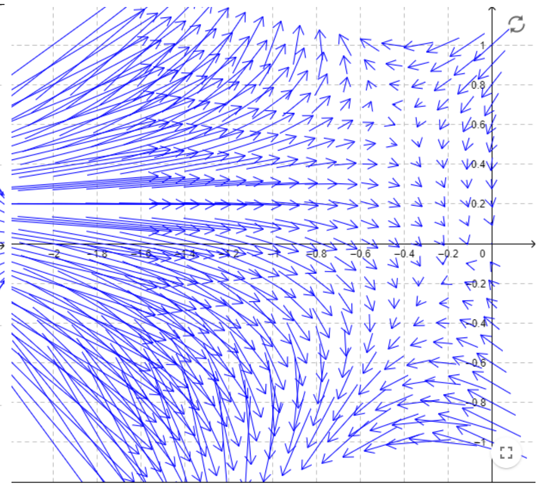
$$\nabla f = \langle 3x^2 - 3y^2 + y, -6xy + x - 2y \rangle$$

$$\begin{gathered} 3x^2 - 3y^2 + y = 0 \\ -6xy + x - 2y = 0 \end{gathered}$$

Where are each of the derivatives equal to zero?

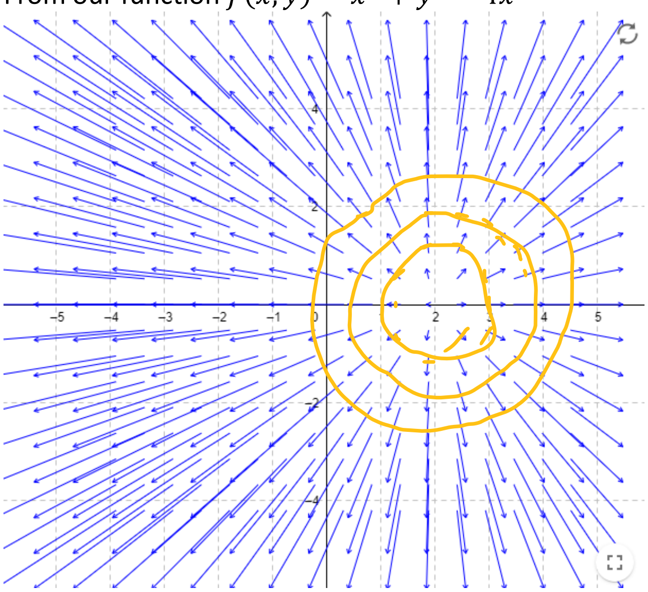


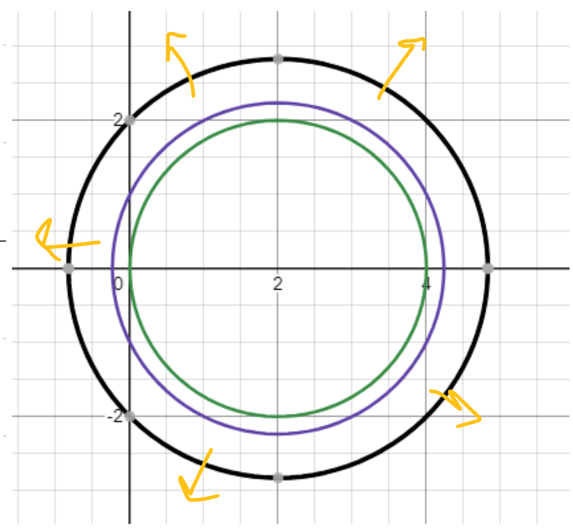




Level curves are always perpendicular to the gradient vectors.

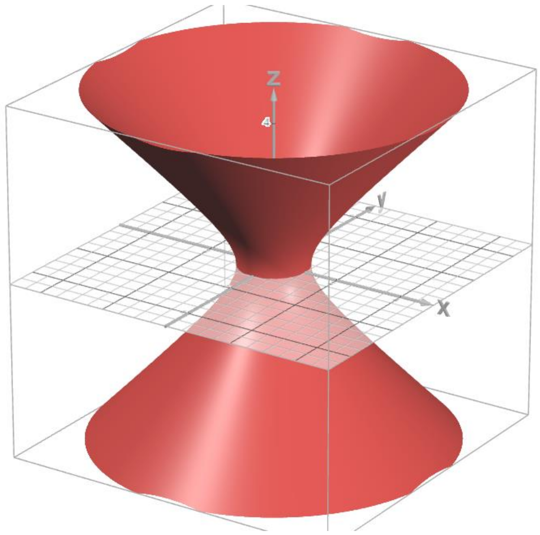
From our function $f(x, y) = x^2 + y^2 - 4x$





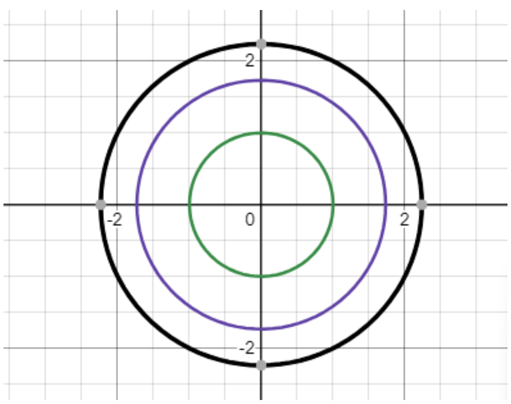
Looking at a graph from the “side”, these are called traces, and typically the values used are axes symmetry or x=0, y=0.

$$x^2 + y^2 - z^2 = 1$$



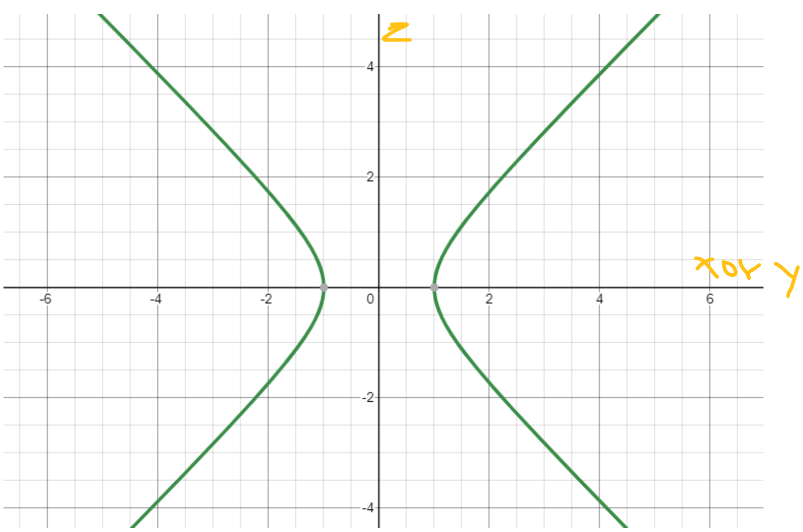
Level curves are when we look down from above and do cross-sectional slices at constant values of z.

$$x^2 + y^2 = c^2 + 1$$



Traces are when I set x=0, or y=0

$$y^2 - z^2 = 1$$
$$x^2 - z^2 = 1$$



Conservative Vector Fields and Potential Functions

Conservative Vector Fields are essentially gradient fields: we don't necessarily know the function used to derive that field. That function that was used to derive field is referred to as a potential function.

Testing for a conservative vector field:
If $\vec{F}$ is conservative, then $\vec{\nabla} \times \vec{F} = 0$

$$f(x,y) = x^2 + y^2 - 4x$$
$$\nabla f = \langle 2x - 4, 2y \rangle$$
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - 4 & 2y & 0 \end{vmatrix} = \langle 0 - 0, -(0 - 0), 0 - 0 \rangle = \vec{0}$$

$$f(x,y,z) = x^2 + xy + z^3 + 2y$$

$$\nabla f = \langle 2x + y, x + 2, 3z^2 \rangle$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + y & x + 2 & 3z^2 \end{vmatrix} = \langle 0 - 0, -(0 - 0), 1 - 1 \rangle = \vec{0}$$

How do we find the potential function:

A one variable integration with a multi-variable function

$$\vec{F} = \langle 2x - 4, 2y \rangle$$

$$\int 2x - 4 \, dx = x^2 - 4x + g(y)$$

$$\int 2y \, dy = y^2 + h(x)$$

$$f(x,y) = x^2 - 4x + y^2 + K$$

$$\vec{F} = \langle 2x + y, x + 2, 3z^2 \rangle$$

$$\int 2x + y \, dx = x^2 + xy + g(y,z)$$

$$\int x + 2 \, dy = xy + 2y + h(x,z)$$

$$\int 3z^2 dz = z^3 + k(x,y)$$

$$f(x, y, z) = xy + x^2 + 2y + z^3 + K$$

Fundamental Theorem of Line Integrals

If the vector field is conservative $\int_C \vec{F} \cdot d\vec{r}$, then the path one takes from point a to point b does not matter for the value of the line integral.

Find the potential function for the field, and plug into starting and stopping points directly into the potential function.

Given $\vec{F} = \nabla f$, then $\int_C \vec{F} \cdot d\vec{r}$ between (x_0, y_0, z_0) and (x_1, y_1, z_1) is $f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$.

If the path is a closed curve (it starts and stops in the same place), the value of the integral is always 0.