6/11/2024

Tangent Planes and Normal Lines – Rectangular (14.4) and Parametric form (16.6)
Arc Length and Curvature (13.3)

Tangent Planes are planes that are tangent to a surface and approximate the surface at the point where they touch. Planes are defined by a vector that is perpendicular to the plane. What we want is a vector that is perpendicular to the surface at a given point. From that perpendicular vector, we can calculate the tangent plane, and the normal line (is a line that is perpendicular to the surface at that point, and perpendicular to the tangent plane).

$$z = f(x, y)$$
$$\nabla f = \langle f_x, f_y \rangle$$

The problem with this, is that the gradient vector is only in two dimensions, and we need a vector that exists in three dimensions.

Transform our function $z = f(x, y)$ into $F(x, y, z)$ by moving everything to one side of the equation. Either:

$$F(x, y, z) = z - f(x, y)$$
$$F(x, y, z) = f(x, y) - z$$

Perpendicular to the plane is the gradient of the big-F function.

$$\nabla F(x, y, z) = \langle -f_x(x, y), -f_y(x, y), 1 \rangle$$

Find the tangent to the surface $z = 2x^2 + y^2$ at the point (1,1,3)

$$F(x, y, z) = z - 2x^2 - y^2$$
$$\nabla F = \langle -4x, -2y, 1 \rangle$$
$$\nabla F(1,1,3) = \langle -4, -2, 1 \rangle$$

Plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Line:

$$\frac{(x - x_0)}{a} = \frac{y - y_0}{b} = \frac{(z - z_0)}{c}$$
$$\vec{r}(t) = \langle at + x_0, bt + y_0, ct + z_0 \rangle$$

Tangent plane:

$$-4(x - 1) - 2(y - 1) + 1(z - 3) = 0$$

Normal line:

$$\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z - 3}{1}$$

$$\vec{r}(t) = \langle -4t + 1, -2t + 1, t + 3 \rangle$$

Example.

$$z = \ln(x - 2y), (3,1,0)$$

Find the equation of the tangent plane and the normal line at the given point.

$$F(x, y, z) = z - \ln(x - 2y)$$
$$\nabla F = \langle -\frac{1}{x - 2y}, \frac{2}{x - 2y}, 1 \rangle$$
$$\nabla F(3,1,0) = \langle -1,2,1 \rangle$$

Tangent plane:

$$-1(x - 3) + 2(y - 1) + 1(z - 0) = 0$$

Normal line:

$$\frac{x - 3}{-1} = \frac{y - 1}{2} = \frac{z}{1}$$

$$\vec{r}(t) = \langle -t + 3, 2t + 1, t \rangle$$

Implicit functions: for the tangent line, we don't need to solve for z first, we can just move everything to one side, and then take the gradient. (However, later, for surface area and other applications, we will need to solve for z first, and make the function explicit.)

Example.
Cone:

$$z^2 = x^2 + y^2$$

Find the tangent plane to cone at the point (3,4,5).

$$F(x, y, z) = z^2 - x^2 - y^2$$
$$\nabla F = \langle -2x, -2y, 2z \rangle$$
$$\nabla F(3,4,5) = \langle -6, -8, 10 \rangle$$

Tangent plane:

$$-6(x - 3) - 8(y - 4) + 10(z - 5) = 0$$

Normal line:

$$\frac{x - 3}{-6} = \frac{y - 4}{-8} = \frac{z - 5}{10}$$

$$\vec{r}(t) = \langle -6t + 3, -8t + 4, 10t + 5 \rangle$$

$$z = \sqrt{x^2 + y^2}$$

$$F(x, y, z) = z - \sqrt{x^2 + y^2}$$
$$\nabla F = \langle -\frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2x), -\frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2y), 1 \rangle = \langle -\frac{x}{\sqrt{x^2 + y^2}}, -\frac{y}{\sqrt{x^2 + y^2}}, 1 \rangle$$

$$\nabla F(3,4,5) = \langle -\frac{3}{5}, -\frac{4}{5}, 1 \rangle$$

Tangent:

$$-\frac{3}{5}(x-3) - \frac{4}{5}(y-4) + 1(z-5) = 0$$

Etc.

Parametric Surfaces.
We still want to find a vector perpendicular to the surface:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Find a tangent plane to the parametric surface at the point (5,2,3)…

The point given in space is typically in (x,y,z) form, and not in terms of u and v. So we will have to do some algebra to get the values for u and v that we will need.

$$\vec{r}_u = \langle x_u(u,v), y_u(u,v), z_u(u,v) \rangle$$
$$\vec{r}_v = \langle x_v(u,v), y_v(u,v), z_v(u,v) \rangle$$

Both of these derivative vectors are in the plane of the tangent. To obtain the perpendicular vector, we need to do the cross product.

Perpendicular vector will be:

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

$\vec{r}(u,v) = \langle u^2+1, v^3+1, u+v \rangle$ at the point (5,2,3)

Find u and v for this point.

$$u^2 + 1 = 5$$
$$u = \pm 2$$
$$v^3 + 1 = 2$$
$$v = 1$$
$$u + v = 3$$
$$u = 2$$

$$(u,v) = (2,1)$$

$$\vec{r}_u = \langle 2u, 0,1 \rangle$$
$$\vec{r}_v = \langle 0,3v^2, 1 \rangle$$

$$\vec{r}_u(2,1) = \langle 4,0,1 \rangle$$
$$\vec{r}_v(2,1) = \langle 0,3,1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 0 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \langle -3,4,12 \rangle$$

Tangent plane:

$$-3(x-5) + 4(y-2) + 12(z-3) = 0$$

Normal line:

$$\frac{x-5}{-3} = \frac{y-2}{4} = \frac{z-3}{12}$$

$$\vec{r}(t) = \langle -3t+5, 4t+2, 12t+3 \rangle$$

Arc Length and Curvature (13.3/3.3)

Recall from parametric equations, that the arc length formula was

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

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$$s = \int_a^b \|\vec{r}\,'(t)\| dt = \int_a^b ds$$

Example. Find the length of the helix on the interval $[0,2\pi], \vec{r}(t) = \langle 3\cos t, 3\sin t, 2t \rangle$

$$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 2 \rangle$$
$$\|\vec{r}\,'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 4} = \sqrt{9+4} = \sqrt{13}$$

$$s = \int_0^{2\pi} \sqrt{13} dt = 2\pi\sqrt{13}$$

Example.

$$\vec{r}(t) = \langle t^2+1, t^3-1, 4t \rangle, [0,3]$$
$$\vec{r}\,'(t) = \langle 2t, 3t^2, 4 \rangle$$
$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 9t^4 + 16}$$

$$s = \int_a^b \|\vec{r}\,'(t)\| dt = \int_0^3 \sqrt{4t^2 + 9t^4 + 16} dt$$

Integrated numerically.

Converting a parametric function into a function of its arc length would be by:

$$s(t) = \int_0^t \|\vec{r}'(u)\| du$$

Consider the helix example from earlier:

the helix on the interval $[0,2\pi], \vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

$$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 2\rangle$$
$$\|\vec{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 4} = \sqrt{9+4} = \sqrt{13}$$

Represent the helix in terms of its arc length:

$$s(t) = \int_0^t \sqrt{13} du = t\sqrt{13}$$

$$s = t\sqrt{13} \to t = \frac{s}{\sqrt{13}}$$

$$\vec{r}(s) = \langle 3\cos\frac{s}{\sqrt{13}}, 3\sin\frac{s}{\sqrt{13}}, \frac{2s}{\sqrt{13}}\rangle$$

Curvature:

$$\kappa = \left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t)\times\vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

Find the curvature of the helix $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle -3\sin t, 3\cos t, 2\rangle}{\sqrt{13}}$$

$$\vec{T}(s) = \frac{\langle -3\sin\frac{s}{\sqrt{13}}, 3\cos\frac{s}{\sqrt{13}}, 2\rangle}{\sqrt{13}}$$

$$\vec{T}'(s) = \frac{1}{\sqrt{13}}\langle -\frac{3}{\sqrt{13}}\cos\frac{s}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\sin\frac{s}{\sqrt{13}}, 0\rangle$$

$$\kappa = \|\vec{T}'(s)\| = \left\|\frac{1}{\sqrt{13}}\langle -\frac{3}{\sqrt{13}}\cos\frac{s}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\sin\frac{s}{\sqrt{13}}, 0\rangle\right\| = \frac{1}{\sqrt{13}}\left(\frac{3}{\sqrt{13}}\right)\left\|\langle -\cos\frac{s}{\sqrt{13}}, -\sin\frac{s}{\sqrt{13}}, 0\rangle\right\|$$

$$= \frac{3}{13}\sqrt{\cos^2\frac{s}{\sqrt{13}} + \sin^2\frac{s}{\sqrt{13}}} = \frac{3}{13}$$

Use the other version of the formula to find the arclength.

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

Find the curvature of the helix $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

$$\vec{r}'(t) = \langle -3\sin t, 3\cos t, 2\rangle$$
$$\vec{r}''(t) = \langle -3\cos t, -3\sin t, 0\rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin t & 3\cos t & 2 \\ -3\cos t & -3\sin t & 0 \end{vmatrix} = \langle 6\sin t, 6\cos t, 9\sin^2 t + 9\cos^2 t\rangle = \langle 6\sin t, 6\cos t, 9\rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\langle 6\sin t, 6\cos t, 9\rangle\| = \sqrt{36\sin^2 t + 36\cos^2 t + 81} = \sqrt{36+81} = \sqrt{117} = 3\sqrt{13}$$

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{3\sqrt{13}}{\left(\sqrt{13}\right)^3} = \frac{3\sqrt{13}}{13\sqrt{13}} = \frac{3}{13}$$

Has a constant curvature.

Find the curvature for

$$\vec{r}(t) = \langle t^2+1, t^3-1, 4t\rangle$$
$$\vec{r}'(t) = \langle 2t, 3t^2, 4\rangle$$
$$\vec{r}''(t) = \langle 2, 6t, 0\rangle$$
$$\|\vec{r}'(t)\| = \sqrt{4t^2 + 9t^4 + 16}$$

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & 3t^2 & 4 \\ 2 & 6t & 0 \end{vmatrix} = \langle -24t, -8, 12t^2 - 6t^2\rangle = \langle -24t, -8, 6t^2\rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \|\langle -24t, -8, 6t^2\rangle\| = \sqrt{576t^2 + 64 + 36t^4}$$

$$\kappa = \frac{\sqrt{576t^2 + 64 + 36t^4}}{\left(\sqrt{4t^2 + 9t^4 + 16}\right)^3}$$

Find the curvature at a particular point, such as $t = 1$

$$\kappa(1) = \frac{\sqrt{676}}{\left(\sqrt{29}\right)^3} \approx 0.166485\ldots$$

The radius of curvature, from the idea approximating the curve at a given point with a circle of a particular radius

$$R \approx \frac{1}{\kappa}$$

$$R = \frac{\sqrt{29^3}}{\sqrt{676}} \approx 6.00653$$

Alternative form for $y = f(x)$.

$$\kappa = \frac{|f''(x)|}{[1 + [f'(x)]^2]^{3/2}}$$

Recall

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

The curve has to be in the plane.

Next time surface area and Green's theorem.