6/12/2024

Surface Area (15.6)
Green's Theorem (16.4)

When we found the arclength, we found the magnitude of the tangent vector and integrated that to obtain the length of the curve.
When we go to surface area (one dimension up from that), we also will use the vector that describes the derivative (for the tangent plane). The relationship between the magnitude of the derivative vector and the measure of the curve/surface is similar.

For the surface are, we are not going to use the gradient of the surface directly, but we will use the version of the gradient that we used to find the tangent plane:

Recall:
Started with a function $f(x,y)=z$, and we created a 3-variable function from that $F(x,y,z)=z-f(x,y)$ and then found the gradient of that function F.

Formula for the surface area:

$$A=\int_a^b\int_{h(x)}^{g(x)}\|\vec{\nabla}F\|dA$$

Recall:

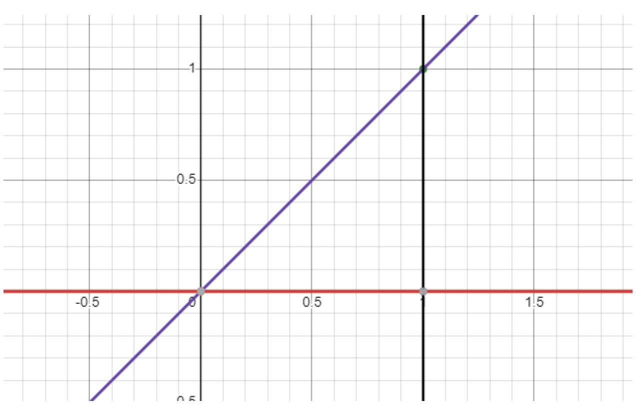
$$\nabla F=\langle -f_x,-f_y,1\rangle$$

That makes the integral:

$$\int_a^b\int_{h(x)}^{g(x)}\|\vec{\nabla}F\|dA=\int_a^b\int_{h(x)}^{g(x)}\sqrt{[f_x]^2+[f_y]^2+1}dydx$$

$f(x,y)$ must be an explicit function of x and y, and cannot be implicit.

Example.
Find the surface area of the part of the surface $z=x^2+2y$ that lies above the triangular region in the xy-plane with vertices (0,0), (1,0), (1,1).



$$F(x,y,z) = z - x^2 - 2y$$
$$\nabla F = \langle -2x, -2, 1 \rangle$$
$$\|\nabla F\| = \sqrt{4x^2 + 4 + 1} = \sqrt{4x^2 + 5}$$

$$A = \int_0^1 \int_0^x \sqrt{4x^2 + 5}\, dy dx = \int_0^1 y\sqrt{4x^2 + 5}\Big|_0^x dx = \int_0^1 x\sqrt{4x^2 + 5}\, dx$$

$$u = 4x^2 + 5, du = 8xdx \to \frac{1}{8} du = xdx$$

$$\int u^{\frac{1}{2}} \left(\frac{1}{8}\right) du = \frac{1}{8}\left(\frac{2}{3}\right) u^{\frac{3}{2}}$$

$$\frac{1}{12}\left(9^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = \frac{27 - 5\sqrt{5}}{12}$$

Example.
Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

$$F(x,y,z) = z - x^2 - y^2$$
$$\nabla F = \langle -2x, -2y, 1 \rangle$$
$$\|\nabla F\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4(x^2 + y^2) + 1} = \sqrt{4r^2 + 1}$$

$$x^2 + y^2 = 9$$

Switch to polar:

$$r^2 = 9, r = 3$$

$$\int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r dr d\theta$$

$$u = 4r^2 + 1, du = 8rdr \to \frac{1}{8} du = rdr$$

$$\int u^{\frac{1}{2}} \left(\frac{1}{8}\right) du = \frac{1}{8}\left(\frac{2}{3}\right) u^{\frac{3}{2}}$$

$$\frac{1}{12} \int_0^{2\pi} \left(37^{\frac{3}{2}} - 1\right) d\theta = \frac{2\pi}{12}(37\sqrt{37} - 1) = \frac{\pi}{6}(37\sqrt{37} - 1)$$

Example.
Find the area of the top half of the sphere $x^2 + y^2 + z^2 = 16$.

$$z = \sqrt{16 - x^2 - y^2}$$
$$F(x,y,z) = z - \sqrt{16 - x^2 - y^2}$$

$$\nabla F = \langle \frac{\left(\frac{1}{2}\right)(-1)(-2x)}{\sqrt{16-x^2-y^2}}, \frac{\left(\frac{1}{2}\right)(-1)(-2y)}{\sqrt{16-x^2-y^2}}, 1\rangle = \langle \frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}}, 1\rangle$$

$$\|\nabla F\| = \sqrt{\frac{x^2}{16-x^2-y^2} + \frac{y^2}{16-x^2-y^2} + \frac{1(16-x^2-y^2)}{16-x^2-y^2}} = \sqrt{\frac{16}{16-x^2-y^2}} = \frac{4}{\sqrt{16-x^2-y^2}}$$

$$x^2 + y^2 = 16$$

$$\|\nabla F\| = \frac{4}{\sqrt{16-r^2}}$$

$$r = 4$$

$$\int_0^{2\pi}\int_0^4 \frac{4}{\sqrt{16-r^2}} r dr d\theta$$

$$u = 16 - r^2, du = -2rdr \to -\frac{1}{2}du = rdr$$

$$\int 4\left(-\frac{1}{2}\right)u^{-\frac{1}{2}}du = -2(2)u^{\frac{1}{2}} = -4\sqrt{16-r^2}$$

$$\int_0^{2\pi} 16 d\theta = 32\pi$$

Surface area of a sphere is $SA = 4\pi r^2$, hemisphere: $SA = 2\pi r^2 = 2\pi(4^2) = 32\pi$

Parametric surfaces:

Want to integrate the magnitude of vector (scaled appropriately) for the normal vector to the surface in order to obtain the surface area.

That normal vector is

$$\vec{r}_u \times \vec{r}_v$$

$$A = \int\int_Q \|\vec{r}_u \times \vec{r}_v\| \, dA$$

Example.
Find the area of the surface $\vec{r}(u, v) = \langle 2u\sin v, 2u\cos v, u^2\rangle$
Over: $0 \le u \le 2, 0 \le v \le 2\pi$

$$\vec{r}_u = \langle 2\sin v, 2\cos v, 2u\rangle$$
$$\vec{r}_v = \langle 2u\cos v, -2u\sin v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\sin v & 2\cos v & 2u \\ 2u\cos v & -2u\sin v & 0 \end{vmatrix} = \langle 0 + 4u^2 \sin v\,, -(0 - 4u^2\cos v), -4u\sin^2 v - 4u\cos^2 v\rangle$$

$$= \langle 4u^2 \sin v\,, 4u^2 \cos v\,, -4u\rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{16u^4\sin^2 v + 16u^4\cos^2 v + 16u^2} = \sqrt{16u^4 + 16u^2} = \sqrt{(16u^2)(u^2+1)} = 4u\sqrt{u^2+1}$$

$$\int_0^{2\pi}\int_0^2 4u\sqrt{u^2+1}\,dudv$$

$$w = u^2 + 1, dw = 2udu$$

$$\int 2w^{\frac{1}{2}}dw = 2\left(\frac{2}{3}\right)w^{\frac{3}{2}} = \frac{4}{3}(u^2+1)^{\frac{3}{2}}$$

$$\int_0^{2\pi} \frac{4}{3}\left(5^{\frac{3}{2}} - 1\right)dv = \frac{4\left(5\sqrt{5}-1\right)}{3}(2\pi) = \frac{8\pi\left(5\sqrt{5}-1\right)}{3}$$

Green's Theorem
A way of calculating some line integrals using the area of the region they enclose.
First requirement: the curve must be closed (start and stop at the same place).
Second "requirement": the field is not conservative
Third requirement: the line integral is the vector field version or the differential version

$$\int_C \vec{F}\cdot d\vec{r} = \int_c Mdx + Ndy$$

For closed curves you'll sometime see the integral notation with a circle on the integral.

$$\oint Mdx + Ndy$$

This doesn't change anything, just it just indicates that the path is closed.

Green's theorem allows us to swap a line integral for an integral over the area of the region enclosed by the curve.

It says:

$$\oint Mdx + Ndy = \iint \left(\frac{\partial N}{dx} - \frac{\partial M}{dy}\right)dA$$

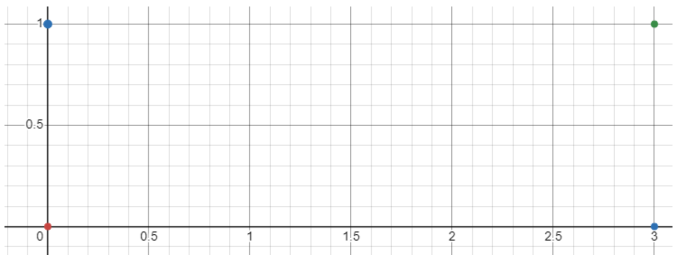
The bounds on the area are defined by the region enclosed by the curve(s).

What happens if the field is conservative?

There exists a potential function F, such that $M = \frac{\partial F}{dx}, N = \frac{\partial F}{dy}$, therefore $\frac{\partial N}{dx} = \frac{\partial^2 F}{\partial x\partial y}, \frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y\partial x}$, since the mixed partials are the same, the function being integrated over the region is 0, so the integral will be zero. This is consistent with the fundamental theorem of line integrals which says that any line integral over a closed curve in a conservative vector field will be zero since the value is independent of path.

This is why we are interested in the cases where the field is not conservative.

Example.
Evaluate $\oint xydx + x^2dy$ over the rectangle with vertices (0,0), (3,0),(3,1),(0,1)



If we were to do this by the definition: we would need 4 paths. From (0,0) to (3,0), and then another from (3,0) to (3,1), and then another from (3,1) to (0,1) and then finally a fourth from (0,1) to (0,0).

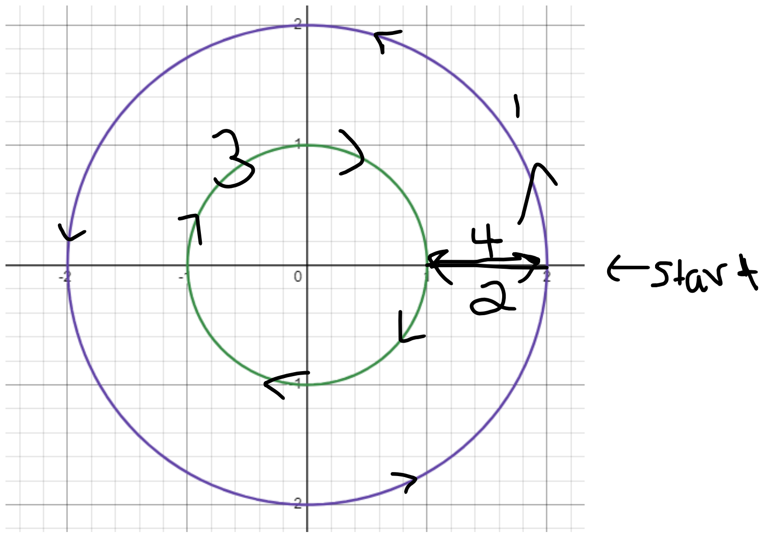
Do all the substitutions, and derivatives and integrate 4 times, and then add everything up.

$$M = xy, N = x^2$$
$$\frac{\partial N}{\partial x} = 2x, \frac{\partial M}{\partial y} = x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x = x$$

$$\int_0^3 \int_0^1 xdydx = \int_0^3 xy|_0^1 dx = \int_0^3 xdx = \frac{1}{2}x^2\Big|_0^3 = \frac{9}{2}$$

What if we wanted to do a line integral over an annulus?
One complication from this is that paths are not obviously connected, so how do we make it fit into our assumptions for green's theorem?



A path like this allows us to connect the dots and travel the entire path and still start and stop in the same place, this allows us to deal with more complex regions than just solid ones.

Evaluate $\oint y^2dx + 3xydy$ where C is the path around the region bounded by the circle of radius 2 and the circle of radius 1 in the xy-plane.

$$\frac{\partial N}{dx} = 3y, \frac{\partial M}{dy} = 2y, \frac{\partial N}{dx} - \frac{\partial M}{dy} = 3y - 2y = y$$

$$\int_0^{2\pi}\int_1^2 r\sin\theta\, rdrd\theta = \int_0^{2\pi}\int_1^2 r^2\sin\theta\, drd\theta = \int_0^{2\pi}\frac{1}{3}r^3\Big|_1^2 \sin\theta\, d\theta = \frac{1}{3}(7)\int_0^{2\pi}\sin\theta\, d\theta =$$

$$\frac{7}{3}(-\cos\theta|_0^{2\pi} = \frac{7}{3}\big(-1-(-1)\big) = 0$$

Example.

Evaluate $\oint 4ydx + 2xdy$ where C is the boundary of the ellipse $x^2 + 2y^2 = 2 \rightarrow \frac{x^2}{2} + y^2 = 1$

$$\frac{\partial N}{\partial x} = 2, \frac{\partial M}{\partial y} = 4, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 4 = -2$$

$$\iint(-2)dA$$

Actually, the value of the integral is just the area of the region times the constant (if the only function I'm integrating is a constant).

$$-2 \times (area\ of\ the\ ellipse)$$

$$A = \pi ab$$
$$a = \sqrt{2}, b = 1, A = \pi\sqrt{2}$$

Therefore, my integral is just $-2\pi\sqrt{2}$

Example.

Evaluate $\oint(1 - y^3)dx + \left(x^3 + e^{y^2}\right)dy$ C is the boundary of the region between the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 9$.

$$\frac{\partial N}{\partial x} = 3x^2, \frac{\partial M}{dy} = -3y^2, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 3x^2 - (-3y^2) = 3x^2 + 3y^2 = 3r^2$$

$$\int_0^{2\pi} \int_2^3 3r^2(r\, drd\theta) = \int_0^{2\pi} \int_2^3 3r^3\, drd\theta$$

Green's theorem hints:

Messy integrals in vector field or differential form that will simplify upon taking a derivative

Description of the path in terms of the boundary of a region

Next time, surface integrals.