6/13/2024

Surface Integrals (16.7)

General Surface area integrals:

$$\iint f(x,y,z)dS = \iint f(r(u,v))\|\vec{r}_u \times \vec{r}_v\|dA = \iint f(x,y,g(x,y))\|\nabla G\|dA$$
$$= \iint f(x,y,g(x,y))\sqrt{1+[f_x]^2+[f_y]^2}dA$$

Example.
Evaluate $\iint ydS$ where S is the surface $z = x + y^2, 0 \le x \le 1, 0 \le y \le 2$

$$G = z - x - y^2$$
$$\nabla G = \langle -1, -2y, 1\rangle$$
$$\|\nabla G\| = \sqrt{1+4y^2+1} = \sqrt{2+4y^2}$$

$$\int_0^1 \int_0^2 y\sqrt{2+4y^2}dydx = \left(\int_0^1 dx\right)\left(\int_0^2 y\sqrt{2+4y^2}dy\right)$$

$$u = 2+4y^2, du = 8ydy, \frac{1}{8}du = ydy$$

$$\int \frac{1}{8}u^{\frac{1}{2}}du = \frac{1}{8}\left(\frac{2}{3}\right)u^{\frac{3}{2}} = \frac{1}{12}(2+4y^2)^{\frac{3}{2}}$$

$$\frac{1}{12}(2+4y^2)^{\frac{3}{2}}\Big|_0^2 = \frac{1}{12}\left[18^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] = \frac{1}{12}[18\sqrt{18} - 2\sqrt{2}] = \frac{1}{6}[9\sqrt{18} - \sqrt{2}] = \frac{27\sqrt{2}-\sqrt{2}}{6} = \frac{26\sqrt{2}}{6}$$
$$= \frac{13\sqrt{2}}{3}$$

Example.
Evaluate $\iint zdS$ where S is the surface whose sides are given by the cylinder $x^2 + y^2 = 1$, whose bottom is given by the disk $x^2 + y^2 \le 1$ on the plane z=0, and whose top is the part of the plane $z = 1 + x$ that lies above the disk.

Parametrize the surface (sides)
Sides: $\vec{r}_1(u,v) = \langle \cos u, \sin u, v\rangle$ $(0 \le u \le 2\pi, 0 \le v \le 1 + \cos u)$
Bottom: $\vec{r}_2(u,v) = \langle v\cos u, v\sin u, 0\rangle$ $(0 \le u \le 2\pi, 0 \le v \le 1)$
Top: $\vec{r}_3(u,v) = \langle u, v, 1+u\rangle$

Sides:

$$\vec{r}_{1_u} = \langle -\sin u, \cos u, 0\rangle$$
$$\vec{r}_{1_v} = \langle 0,0,1\rangle$$

$$\vec{r}_{1_u} \times \vec{r}_{1_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u\,, \sin u\,, 0 \rangle$$

$$\left\| \vec{r}_{1_u} \times \vec{r}_{1_v} \right\| = \sqrt{\cos^2 u + \sin^2 u} = \sqrt{1} = 1$$

$$\int_0^{2\pi} \int_0^{1+\cos u} v(1)\,dv\,du = \int_0^{2\pi} \frac{1}{2} v^2 \Big|_0^{1+\cos u} du = \frac{1}{2} \int_0^{2\pi} (1 + \cos u)^2 du$$

$$= \frac{1}{2} \int_0^{2\pi} 1 + 2\cos u + \cos^2 u\, du = \frac{1}{2} \int_0^{2\pi} 1 + 2\cos u + \frac{1}{2}(1 + \cos 2u) du =$$

$$\frac{1}{2} \int_0^{2\pi} 1 + 2\cos u + \frac{1}{2} + \frac{1}{2}\cos 2u\, du = \frac{1}{2} \int_0^{2\pi} \frac{3}{2} + 2\cos u + \frac{1}{2}\cos 2u\, du$$

$$= \frac{1}{2}\left[\frac{3}{2}u + 2\sin u + \frac{1}{4}\sin 2u\right]_0^{2\pi} = \frac{1}{2}(3\pi) = \frac{3\pi}{2}$$

Bottom:
On the bottom, z=0

$$\int_0^{2\pi} \int_0^1 0 \left\| \vec{r}_{2_u} \times \vec{r}_{2_v} \right\| dv\,du = 0$$

The value of dS does not matter since the function is multipled by 0, the whole integral is zero regardless.

Top:
Could use the parametrization in terms of u and v, or since z is an explicit function of x and y already, we can use the rectangular coordinate method.

$$G = z - 1 - x$$
$$\nabla G = \langle -1{,}0{,}1 \rangle$$
$$\|\nabla G\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\iint z\,dS = \iint (1 + x)\|\nabla G\|\,dA = \int_0^{2\pi} \int_0^1 (1 + r\cos\theta)\sqrt{2}\,r\,dr\,d\theta$$

Using rectangular and then switching to polar requires less algebra than working with the parametrization (no cross product involved).

$$\sqrt{2} \int_0^{2\pi} \int_0^1 r + r^2 \cos\theta\, dr\, d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{1}{3}r^3 \cos\theta\right]_0^1 d\theta = \sqrt{2} \int_0^{2\pi} \left[\frac{1}{2} + \frac{1}{3}\cos\theta\right] d\theta$$

$$= \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{3}\sin\theta\right]_0^{2\pi} = \sqrt{2}[\pi] = \pi\sqrt{2}$$

Add all the pieces together:

$$\frac{3\pi}{2} + 0 + \pi\sqrt{2} = \pi\left(\frac{3}{2} + \sqrt{2}\right)$$

Orientable surfaces

All of surfaces that we will be dealing with will be orientable surfaces, however, there are some surfaces that are not orientable and would be able to be used in the applications to follow.

Non-orientable surface example is the Moebius strip.

Closed surface orientability

We want the normal to the surface to be facing outward for a positive orientation, and inward for a negative orientation.

Terminology:

When the surface integral (over a vector field) is positive, say that the field/sink is a source.

When the surface integral (over a vector field) is negative, we say that it is a sink

When the surface integral (over a vector field) is zero, we say that the flow is incompressible.

Surface integrals over a vector field:

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{n} dS = \iint \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \iint \vec{F} \cdot \nabla G \, dA$$

$$dS = \|\vec{r}_u \times \vec{r}_v\| dA, \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

$$dS = \|\nabla G\| dA, \vec{n} = \frac{\nabla G}{\|\nabla G\|}$$

Sometimes called flux integrals

Example.

Find the flux of the vector field $F(x, y, z) = \langle z, y, x\rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Parametrize:

$$\vec{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v\rangle, 0 \le u \le 2\pi, 0 \le v \le \pi$$

$$F(u, v) = \langle \cos v, \sin u \sin v, \cos u \sin v\rangle$$

$$\vec{r}_u = \langle -\sin u \sin v, \cos u \sin v, 0\rangle$$
$$\vec{r}_v = \langle \cos u \cos v, \sin u \cos v, -\sin v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin u \sin v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{vmatrix} =$$

$$\langle -\cos u \sin^2 v\,, -(\sin u \sin^2 v), -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v\rangle =$$

$$\langle -\cos u \sin^2 v\,, -\sin u \sin^2 v\,, -\sin v \cos v\rangle$$

$$\int_0^{2\pi}\int_0^{\pi} \langle \cos v\,, \sin u \sin v\,, \cos u \sin v\rangle \cdot \langle -\cos u \sin^2 v\,, -\sin u \sin^2 v\,, -\sin v \cos v\rangle dv du =$$

$$\int_0^{2\pi}\int_0^{\pi} -\cos u \cos v \sin^2 v - \sin^2 u \sin^3 v - \cos u \sin^2 v \cos v \; dv du =$$

$$\int_0^{2\pi}\int_0^{\pi} -2\cos u \cos v \sin^2 v - \sin^2 u \sin^3 v \; dv du =$$

$$\int_0^{2\pi}\int_0^{\pi} -2\cos u \cos v \sin^2 v - \sin^2 u \sin(v)\,(1-\cos^2 v) dv du =$$

$$\int_0^{2\pi}\int_0^{\pi} -2\cos u \cos v \sin^2 v - \sin^2 u \sin(v) + \sin^2 u \sin v \cos^2 v \; dv du =$$

$$\int_0^{2\pi} -2\cos u \left(\frac{1}{3}\right)\sin^3 v + \sin^2 u \cos v - \sin^2 u \left(\frac{1}{3}\right)\cos^3 v \Big|_0^{\pi} \; du =$$

$$\int_0^{2\pi} 0 + \sin^2 u\,(-1) - \sin^2 u\,(1) - \frac{1}{3}\sin^2 u\,(-1) + \frac{1}{3}\sin^2 u\,(1) du = \left(-2 + \frac{2}{3}\right)\int_0^{2\pi} \sin^2 u \; du =$$

$$-\frac{4}{3}\left(\frac{1}{2}\right)\int_0^{2\pi} 1 - \cos 2u \; du = -\frac{2}{3}\left[u - \frac{1}{2}\sin 2u\right]_0^{2\pi} = -\frac{2}{3}(2\pi) = -\frac{4\pi}{3}$$

Next time: Divergence Theorem and maybe Stokes' Theorem