6/17/2024

Divergence Theorem (16.9)
Stokes' Theorem (16.8)

The Divergence Theorem:

$$\iint \vec{F} \cdot dS = \iiint \vec{\nabla} \cdot \vec{F}\, dV = \iiint div\, \vec{F}\, dV$$

Does the require that the region bounded by the surface(s) be a closed region.

Example.
Find the flux of the vector field $\vec{F} = \langle z, y, x \rangle$ over the unit sphere.

$$\vec{\nabla} \cdot \vec{F} = 0 + 1 + 0 = 1$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 (1)\rho^2 \sin\phi\, d\rho d\phi d\theta \int_0^{2\pi} \int_0^{\pi} \int_0^1 (1) dV = \frac{4}{3}\pi$$

Example.
Evaluate the flux integral where $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$ and S is the surface of the region bounded by the parabolic cylinder $z = 1 - x^2$ and the planes z=0, y=0 and $y + z = 2$.

To do this using the definition of the flux integral:4 surfaces. We would have to find the normal to the surface for each one. Each one would be a double integral, not necessarily great functions, and each would be over a different set of variables.

Using the divergence theorem:

$$\nabla \cdot \vec{F} = y + 2y + 0 = 3y$$

$$\int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dydzdx = \frac{3}{2}\int_{-1}^{1} \int_0^{1-x^2} y^2\Big|_0^{2-z} dzdx = \frac{3}{2}\int_{-1}^{1} \int_0^{1-x^2} 4 - 4z + z^2\, dzdx =$$

$$\frac{3}{2}\int_{-1}^{1} 4z - 2z^2 + \frac{1}{3}z^3\Big|_0^{1-x^2} dx = \frac{3}{2}\int_{-1}^{1} 4(1-x^2) - 2(1-x^2)^2 + \frac{1}{3}(1-x^2)^3 dx =$$

$$\frac{3}{2}\int_{-1}^{1} 4 - 4x^2 - 2(1 - 2x^2 + x^4) + \frac{1}{3}(1 - 3x^2 + 3x^4 - x^6) dx =$$

$$\frac{3}{2}\int_{-1}^{1} 4 - 4x^2 - 2 + 4x^2 - 2x^4 + \frac{1}{3} - x^2 + x^4 - \frac{x^6}{3} dx = \frac{3}{2}\int_{-1}^{1} \frac{7}{3} - x^2 - x^4 - \frac{x^6}{3} dx =$$

$$3\int_0^1 \frac{7}{3} - x^2 - x^4 - \frac{x^6}{3} dx = 3\left[\frac{7}{3}x - \frac{1}{3}x^3 - \frac{1}{5}x^5 - \frac{x^7}{21}\right]_0^1 = 3\left[\frac{7}{3} - \frac{1}{3} - \frac{1}{5} - \frac{1}{21}\right] = \frac{184}{35}$$

Example.

Find the flux through the surfaces bounding the box enclosed by the planes x=0, x=2, y=0, y=4, z=0, z=3 for the field $\vec{F} = \langle x^2yz, xy^2z, xyz^2\rangle$

If we were to do this by the original definition of the surface integral, we would need six integrals and six normal vectors.

For the divergence theorem:

$$\nabla \cdot \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\int_0^2 \int_0^4 \int_0^3 6xyzdzdydx = \int_0^2 \int_0^4 3xyz^2|_0^3 dydx = \int_0^2 \int_0^4 27xydydx = \int_0^2 \frac{27}{2}xy^2\Big|_0^4 dx =$$

$$\int_0^2 216xdx = 108x^2|_0^2 = 432$$

Divergence theorem converts a surface integral over a closed region into a triple integral over a volume.

Stokes' Theorem

A method of line integrals in 3-space of a curve that represents the boundary of some surface. You can also think of it as the 3D extension of green's theorem to the 3D space, and not just on a plain. The problems where you are expected to apply Stokes' Theorem will use boundaries of surfaces in the problem description rather than planar regions.

$$\int_c \vec{F} \cdot d\vec{r} = \iint \nabla \times \vec{F} \cdot d\vec{S} = \iint curl\, \vec{F} \cdot d\vec{S}$$

Example.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ (orient C to be counterclockwise when viewed from above)

To do this by the definition, we would need to come up with a parametrization of the curve. $\vec{r}(t) = \langle \cos t, \sin t, 2 - \sin t\rangle$. Substitute into the field for x, y, z . Take the derivative of the path. Dot with the field. Then integrate a bunch of trig functions.

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(-y^2)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y^2)\right)\hat{k}$$

$$= \langle 0,0,1 + 2y\rangle$$

Make the choice of surface that will make for the easiest math. Here, choose the plane $y + z = 2$ to be the surface.

$$\int_{\square}^{\square}\int_{\square}^{\square}\langle 0,0,1+2y\rangle \cdot d\vec{S} = \int_{\square}^{\square}\int_{\square}^{\square}\langle 0,0,1+2y\rangle \cdot \langle 0,1,1\rangle dA$$

$$d\vec{S} = \nabla G, G = y + z - 2, \nabla G = \langle 0,1,1\rangle$$

$$\int_{\square}^{\square}\int_{\square}^{\square}(1+2y)dA = \int_{0}^{2\pi}\int_{0}^{1}(1+2r\sin\theta)rdrd\theta = \int_{0}^{2\pi}\int_{0}^{1} r + 2r^2\sin\theta\, drd\theta =$$

$$\int_{0}^{2\pi}\frac{1}{2}r^2 + \frac{2}{3}r^3\sin\theta\Big|_{0}^{1} d\theta = \int_{0}^{2\pi}\frac{1}{2} + \frac{2}{3}\sin\theta\, d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta\Big|_{0}^{2\pi} = \pi$$

Example.
Use Stokes' Theorem to evaluate the line integral over the field $\vec{F} = \langle x^2z^2, y^2z^2, xyz\rangle$ where S is the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$ oriented upward.
Intersects on the plane z=4.

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z^2 & y^2z^2 & xyz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(y^2z^2)\right)\hat{\imath} - \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(x^2z^2)\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}(y^2z^2) - \frac{\partial}{\partial y}(x^2z^2)\right)\hat{k} =$$

$$\langle xz - 2y^2z, -(yz - 2x^2z), 0 - 0\rangle = \langle xz - 2y^2z, 2x^2z - yz, 0\rangle$$

$$G = z - x^2 - y^2$$
$$\nabla G = \langle -2x, -2y, 1\rangle$$

$$\iint \langle xz - 2y^2z, 2x^2z - yz, 0\rangle \cdot \langle -2x, -2y, 1\rangle dA = \iint -2x^2z + 4xy^2z - 4x^2yz + 2y^2z dA$$

Replace one variable with an expression for the other variables.
Replace z with the paraboloid surface.

$$\iint -2x^2(x^2+y^2) + 4xy^2(x^2+y^2) - 4x^2y(x^2+y^2) + 2y^2(x^2+y^2)dA =$$

$$\iint (x^2+y^2)(-2x^2 + 4xy^2 - 4x^2y + 2y^2)dA$$

$$\int_{0}^{2\pi}\int_{0}^{2} r^2(-2r^2\cos^2\theta + 4r\cos\theta\ r^2\sin^2\theta - 4r^2\cos^2\theta\, r\sin\theta + 2r^2\sin^2\theta)r\, drd\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{2} -2r^5\cos^2\theta + 4r^6\cos\theta\sin^2\theta - 4r^6\cos^2\theta\sin\theta + 2r^5\sin^2\theta\, drd\theta =$$

$$\int_0^{2\pi} -\frac{2}{6}r^6\cos^2\theta + \frac{4}{7}r^7\cos\theta\sin^2\theta - \frac{4}{7}r^7\cos^2\theta\sin\theta + \frac{2}{6}r^6\sin^2\theta\Big|_0^2 \, d\theta =$$

$$\int_0^{2\pi} -\frac{64}{3}\cos^2\theta + \frac{512}{7}\cos\theta\sin^2\theta - \frac{512}{7}\cos^2\theta\sin\theta + \frac{64}{3}\sin^2\theta \, d\theta$$

$$\int_0^{2\pi} \left(-\frac{64}{3}\right)(\cos^2\theta - \sin^2\theta) + \frac{512}{7}(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta)d\theta =$$

$$\int_0^{2\pi} \left(-\frac{64}{3}\right)(\cos 2\theta) + \frac{512}{7}(\cos\theta\sin^2\theta - \cos^2\theta\sin\theta)d\theta$$

$$-\frac{64}{3}\left(\frac{1}{2}\right)\sin 2\theta + \frac{512}{7}\left(\frac{1}{3}\sin^3\theta + \frac{1}{3}\cos^3\theta\right)\Big|_0^{2\pi} = 0$$

Consider choosing a different surface with the same boundary:

Suppose you choose the disk of radius 2 at the height z=4.
What is the normal to the surface with an upward orientation?

$$\hat{k} = \langle 0,0,1\rangle$$

How does that change our math?

It doesn't change the curl, but it does change the result of dotting with the curl.

$$\nabla \times F = \langle xz - 2y^2z, 2x^2z - yz, 0\rangle$$

Normal vector dotted with the curl:

$$\langle xz - 2y^2z, 2x^2z - yz, 0\rangle \cdot \langle 0,0,1\rangle = 0 + 0 + 0 = 0$$

The function to be integrated over the region is just 0…. Which means the integral is 0.