6/25/2024

Change of Variables/Jacobians (15.10)
Velocity and Acceleration (13.4)

Jacobian
A matrix of the transformation from one coordinate system to another coordinate system

Two-variable case:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\right) - \frac{\partial y}{\partial u}\left(\frac{\partial x}{\partial v}\right)$$

When we replace dA in our integral for dydx, dxdy, the replacement is given by

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dvdu$$

Think about the polar coordinate transformation:

$$x = r\cos\theta\,, y = r\sin\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \cos\theta\,(r\cos\theta) + \sin\theta\,(r\sin\theta) = r\cos^2\theta + r\sin^2\theta = r$$

$$dA = |r|drd\theta = rdrd\theta$$

Three variable case:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$dV = dzdydx = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

Think about spherical transformations:

$$x = \rho\cos\theta\sin\phi\,, y = \rho\sin\theta\sin\phi\,, z = \rho\cos\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} =$$

$$\cos\theta\sin\phi\,(\rho\cos\theta\sin\phi\,(-\rho\sin\phi) - 0) \\ -(-\rho\sin\theta\sin\phi)(\sin\theta\sin\phi\,(-\rho\sin\phi) - \cos\phi\,\rho\sin\theta\cos\phi) \\ +\rho\cos\theta\cos\phi\,(0 - \cos\phi\,\rho\cos\theta\sin\phi) =$$

$$-\rho^2\cos^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin\phi\cos^2\phi - \rho^2\cos^2\theta\cos^2\phi\sin\phi =$$

$$\rho^2(-\sin^3\phi)(\cos^2\theta + \sin^2\theta) - \rho^2\sin\phi\cos^2\phi\,(\sin^2\theta + \cos^2\theta) =$$

$$-\rho^2\sin\phi\,(\sin^2\phi + \cos^2\phi) = -\rho^2\sin\phi$$

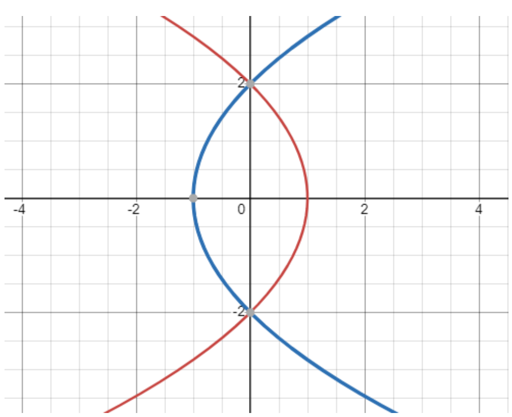
$$dV = dzdydz = |-\rho^2\sin\phi|d\rho d\theta d\phi = \rho^2\sin\phi\,d\rho d\theta d\phi$$

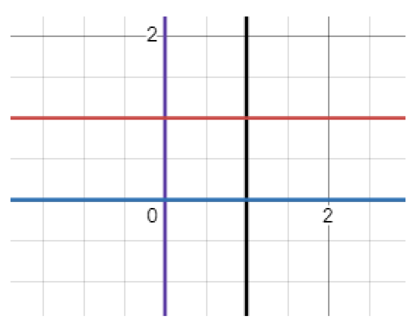
Example.
Evaluate the integral $\iint ydA$ over the region bounded by the parabolas $y^2 = 4 - 4x, y^2 = 4 + 4x$

Use the change of variables $x = u^2 - v^2, y = 2uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$$

$$\iint 2uv(4u^2 + 4v^2)dudv = \int_0^1\int_0^1 2uv(4u^2 + 4v^2)dudv$$





$$\int_0^1\int_0^1 2uv(4u^2 + 4v^2)dudv = \int_0^1\int_0^1 8u^3v + 8uv^3dudv = \int_0^1 [2u^4v + 4u^2v^3]_0^1 dv =$$

$$\int_0^1 2v + 4v^3dv = v^2 + v^4|_0^1 = 1 + 1 = 2$$

Check the value:

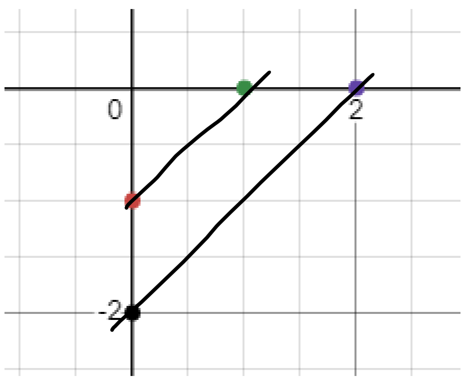
$$y^2 = 4 - 4x$$
$$y^2 - 4 = -4x$$
$$1 - \frac{1}{4}y^2 = x$$

$$y^2 = 4 + 4x$$
$$y^2 - 4 = 4x$$
$$\frac{1}{4}y^2 - 1 = x$$

$$\int_{-2}^{2}\int_{\frac{1}{4}y^2-1}^{1-\frac{1}{4}y^2} y\,dx\,dy$$

Example.

Evaluate $\iint e^{\frac{x+y}{x-y}}dA$ where the region is the trapezoid with vertices $(1,0), (2,0), (0,-2), (0,-1)$



(1,0), (0,-1)

$$m = \frac{1}{1} = 1$$
$$y = x - 1$$

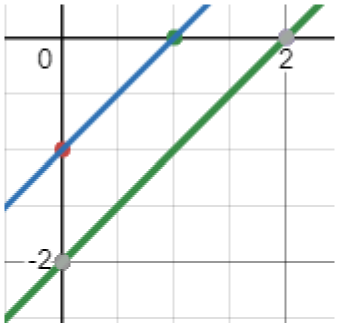
(2,0),(0,-2)

$$m = \frac{2}{2} = 1$$
$$y = x - 2$$

Based on the function, it looks like x+y, and x-y would be useful as substitutions for the function, can we make our lines look like one of these?

$$y - x = -1$$
$$y - x = -2$$

$$x - y = 1$$
$$x - y = 2$$



Let u=x-y [1,2]
If I let v=x+y, I need to find the limits in v

One method is to solve for x and y in terms of u and v.

$$u+v=x-y+x+y=2x$$
$$x=\frac{1}{2}(u+v)$$
$$u-v=x-y-(x+y)=x-y-x-y=-2y$$
$$y=-\frac{1}{2}(u-v)=\frac{1}{2}(v-u)$$

$$v=\left(\frac{1}{2}\right)(u+v)+\left(\frac{1}{2}\right)(v-u)=\frac{1}{2}u+\frac{1}{2}v+\frac{1}{2}v-\frac{1}{2}u=v$$

$$u=\frac{1}{2}(u+v)-\left(\frac{1}{2}\right)(v-u)=u$$

Limits in v are [-u,u]

Find the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\int_1^2\int_{-u}^{u}e^{\frac{v}{u}}\left(\frac{1}{2}\right)dvdu$$

One potential issue, if this does not integrate all the way, you can try switching the transformation (to let u=x+y, v=x-y instead).

$$\frac{1}{2}\int_1^2 e^{\frac{v}{u}}(u)\Big|_{-u}^{u}du=\frac{1}{2}\int_1^2 ue-\frac{u}{e}du=\frac{1}{2}\left[\frac{1}{2}u^2e-\frac{1}{2e}u^2\right]_1^2=\frac{1}{2}[2e-\frac{2}{e}-\frac{e}{2}+\frac{1}{2e}=\frac{1}{2}\left(\frac{3e}{2}-\frac{3}{2e}\right)$$
$$=\frac{3e}{4}-\frac{3}{4e}$$

Velocity and Acceleration on space curves

Example.
A moving particle has an initial position of $\langle 1,0,0\rangle = \vec{r}(0)$, and an initial velocity of $\vec{v}(0) = \langle 1,-1,1\rangle$. If its acceleration is given by $\vec{a}(t) = \langle 4t, 6t, 1\rangle$, find it's velocity vector and position vector at any time t.

$$\vec{v}(t) = \int \vec{a}(t)dt = \int \langle 4t, 6t, 1\rangle dt = \langle 2t^2 + C_1, 3t^2 + C_2, t + C_3\rangle$$

$$2(0)^2 + C_1 = 1 \to C_1 = 1$$
$$3(0)^2 + C_2 = -1 \to C_2 = -1$$
$$0 + C_3 = 1 \to C_3 = 1$$

$$\vec{v}(t) = \langle 2t^2 + 1, 3t^2 - 1, t + 1\rangle$$

$$\vec{r}(t) = \int \vec{v}(t)dt = \int \langle 2t^2 + 1, 3t^2 - 1, t + 1\rangle dt = \langle \frac{2}{3}t^3 + t + C_1, t^3 - t + C_2, \frac{1}{2}t^2 + t + C_3\rangle$$

$$\frac{2}{3}(0)^3 + 0 + C_1 = 1 \to C_1 = 1$$
$$(0)^3 - 0 + C_2 = 0 \to C_2 = 0$$
$$\frac{1}{2}(0)^2 + 0 + C_3 = 0 \to C_3 = 0$$

$$\vec{r}(t) = \langle \frac{2}{3}t^3 + t + 1, t^3 - t, \frac{1}{2}t^2 + t\rangle$$

Projectile motion:

$$\vec{r}(t) = \langle (v_0 \cos\alpha)t, (v_0 \sin\alpha)t - \frac{1}{2}gt^2\rangle$$

α is the angle at which the projectile is launched
g is the gravity constant.
This formula assumes you are launching from the ground, at h_0 if launching from a position above the ground.

Next time, last section on center of mass.