6/26/2024

Centers of Mass (15.7/5.6)
(brief foray into continuous probability distributions—see pg. 517+ in the online book)

Two dimensional case

Total Mass:

$$M = \int_a^b \int_{f(x)}^{g(x)} \rho(x,y)dydx$$

Moment of Mass in the x-direction (moment from the y-axis)

$$M_y = \int_a^b \int_{f(x)}^{g(x)} x\rho(x,y)dydx$$

Moment of Mass in the y-direction (moment from the x-axis)

$$M_x = \int_a^b \int_{f(x)}^{g(x)} y\rho(x,y)dydx$$

Center of Mass:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

From calc 2: suppose that the mass density is constant.

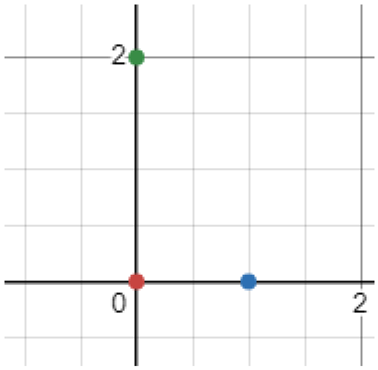
$$M_x = \rho \int_a^b \int_{f(x)}^{g(x)} ydydx$$

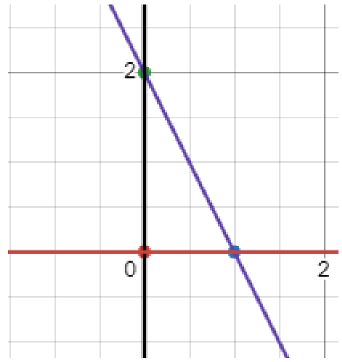
Integrate this one time:

$$\rho \int_a^b \frac{1}{2} y^2 \Big|_{f(x)}^{g(x)} dx = \frac{\rho}{2} \int_a^b [g(x)]^2 - [f(x)]^2 \, dx$$

Which is the formula from Calc 2.

Example.
Find the mass and center of mass of a triangular lamina with vertices (0,0), (1,0), (0,2) if the density function is $\rho(x,y) = 1 + 3x + y$





Lower bound is y=0, upper bound

$$m = \frac{2-0}{0-1} = -2$$
$$y = -2x + 2$$

$$M = \int_0^1 \int_0^{2-2x} 1 + 3x + y \, dydx = \int_0^1 y + 3xy + \frac{1}{2}y^2 \Big|_0^{2-2x} dx =$$

$$\int_0^1 (2-2x) + 3x(2-2x) + \frac{1}{2}(2-2x)^2 \, dx = \int_0^1 2 - 2x + 6x - 6x^2 + \frac{1}{2}(4 - 8x + 4x^2) \, dx =$$

$$\int_0^1 4 - 4x^2 dx = 4x - \frac{4}{3}x^3 \Big|_0^1 = 4 - \frac{4}{3} = \frac{8}{3}$$

$$M_y = \int_0^1 \int_0^{2-2x} x(1 + 3x + y) dydx = \int_0^1 \int_0^{2-2x} x + 3x^2 + xy \, dy \, dx =$$

$$\int_0^1 xy + 3x^2y + \frac{x}{2}y^2 \Big|_0^{2-2x} dx = \int_0^1 x(2-2x) + 3x^2(2-2x) + \frac{x}{2}(2-2x)^2 \, dx =$$

$$\int_0^1 2x - 2x^2 + 6x^2 - 6x^3 + 2x(1 - 2x + x^2) \, dx = \int_0^1 2x - 2x^2 + 6x^2 - 6x^3 + 2x - 4x^2 + 2x^3 \, dx$$

$$= \int_0^1 4x - 4x^3 dx = [2x^2 - x^4]_0^1 = 2 - 1 = 1$$

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\frac{8}{3}} = 1 \times \left(\frac{3}{8}\right) = \frac{3}{8}$$

$$M_x = \int_0^1 \int_0^{2-2x} y(1 + 3x + y) dydx = \int_0^1 \int_0^{2-2x} y + 3xy + y^2 \, dydx =$$

$$\int_0^1 \frac{1}{2}y^2 + \frac{3x}{2}y^2 + \frac{1}{3}y^3 \Big|_0^{2-2x} dx = \int_0^1 \frac{1}{2}(2-2x)^2 + \frac{3x}{2}(2-2x)^2 + \frac{1}{3}(2-2x)^3 \, dx =$$

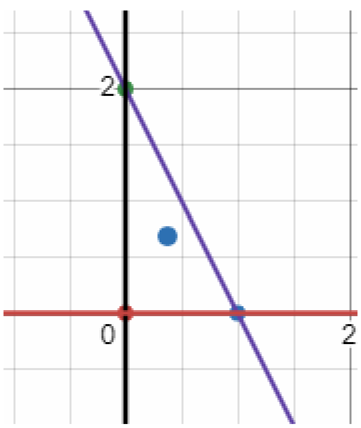
$$\int_0^1 \frac{1}{2}(4-8x+4x^2)+\frac{3x}{2}(4-8x+4x^2)+\frac{1}{3}(8-24x+24x^2-8x^3)\ dx =$$

$$\int_0^1 2-4x+2x^2+6x-12x^2+6x^3+\frac{8}{3}-8x+8x^2-\frac{8}{3}x^3 dx =$$

$$\int_0^1 \frac{14}{3}-6x-2x^2+\frac{10}{3}x^3 dx = \frac{14}{3}x-3x^2-\frac{2}{3}x^3+\frac{10}{12}x^4\Big|_0^1 = \frac{14}{3}-3-\frac{2}{3}+\frac{5}{6}=\frac{11}{6}$$

$$\bar{y}=\frac{M_x}{M}=\frac{\left(\frac{11}{6}\right)}{\frac{8}{3}}=\frac{11}{6}\times\frac{3}{8}=\frac{33}{48}$$

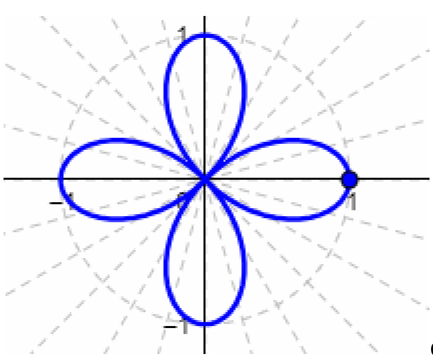
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$
$$2^3(1-x)^3 = 8(1-3x+3x^2-x^3)$$



When the region doesn't have holes or U-shapes, then the center must be inside the region.

Example.

Find the mass and center of mass of the region enclosed by the right loop of the 4-leaf rose $r=\cos 2\theta$ with density function $\rho(x,y)=x^2+y^2$ (i.e. $\rho(r,\theta)=r^2$).



$\cos 2\theta = 0, 2\theta = \frac{\pi}{2}, -\frac{\pi}{2} \rightarrow \theta = \frac{\pi}{4}, -\frac{\pi}{4}$

$$M = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^2 r dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^3 dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{4} r^4 \Big|_{0}^{\cos 2\theta} d\theta = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^4 2\theta \, d\theta =$$

$$\frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2}(1+\cos 4\theta)\right]^2 d\theta = \frac{1}{16}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + 2\cos 4\theta + \cos^2 4\theta \, d\theta =$$

$$\frac{1}{16}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + 2\cos 4\theta + \frac{1}{2}(1+\cos 8\theta) d\theta = \frac{1}{16}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 + 2\cos 4\theta + \frac{1}{2} + \frac{1}{2}\cos 8\theta \, d\theta =$$

$$\frac{1}{16}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{3}{2} + 2\cos 4\theta + \frac{1}{2}\cos 8\theta \, d\theta = \frac{1}{16}\left[\frac{3}{2}\theta + \frac{1}{2}\sin 4\theta + \frac{1}{16}\sin 8\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$

$$\frac{1}{16}\left[\frac{3}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)\right] = \frac{3}{32}\left(\frac{\pi}{2}\right) = \frac{3\pi}{64}$$

To set up the two moments:

$$M_y = \iint x\rho(x,y)dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^3(r\cos\theta) dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^4 \cos\theta \, dr d\theta$$

$$M_x = \iint y\rho(x,y)dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^3(r\sin\theta) dr d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{0}^{\cos 2\theta} r^4 \sin\theta \, dr d\theta$$

To integrate the combination of $\cos 2\theta$ and $\sin\theta$ or $\cos\theta$, apply a double angle formula to get back to a single θ for all expressions.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

The coordinates of the center will be in rectangular coordinates.

In three-dimensions:

$$M = \int_{a}^{b}\int_{h(x)}^{k(x)}\int_{f(x,y)}^{g(x,y)} \rho(x,y,z) dz dy dx$$

Moments of mass are from a plane

Moment of mass from the yz-plane, is the moment of mass in the x-direction

$$M_{yz} = \int_{a}^{b}\int_{h(x)}^{k(x)}\int_{f(x,y)}^{g(x,y)} x\rho(x,y,z) dz dy dx$$

Moment of mass from the xz-plane is the moment of mass in the y-direction

$$M_{xz} = \int_a^b \int_{h(x)}^{k(x)} \int_{f(x,y)}^{g(x,y)} y\rho(x,y,z)dzdydx$$

Moment of mass from the xy-plane is the moment of mass in the z-direction

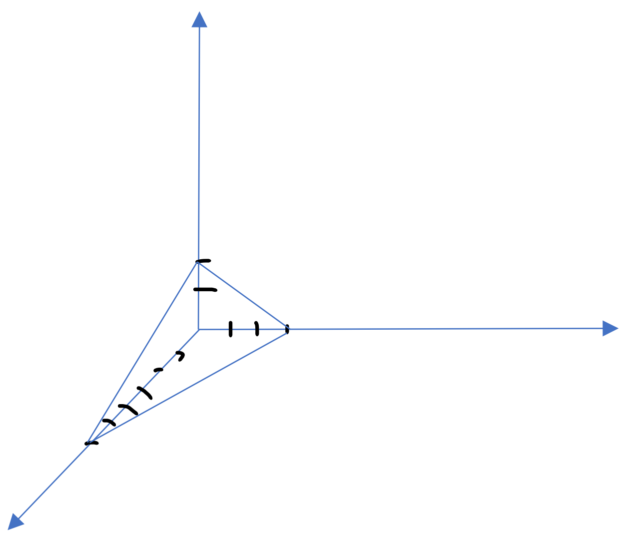
$$M_{xy} = \int_a^b \int_{h(x)}^{k(x)} \int_{f(x,y)}^{g(x,y)} z\rho(x,y,z)dzdydx$$

Center of mass:

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M},\frac{M_{xz}}{M},\frac{M_{xy}}{M}\right)$$

Example.

Find the mass and center of mass for the region bounded by the plane $x + 2y + 3z = 6$ and the coordinate planes. With a mass density function of $\rho(x,y,z) = x^2yz$



$$x + 2y + 3z = 6$$
$$3z = 6 - x - 2y$$
$$z = 2 - \frac{1}{3}x - \frac{2}{3}y$$

$$x + 2y + 3(0) = 6$$
$$x + 2y = 6$$
$$2y = 6 - x$$
$$y = 3 - \frac{1}{2}x$$

$$M = \int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} x^2yz\,dzdydx$$

$$M_{yz} = \int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} x^3yz\,dzdydx$$

$$M_{xz} = \int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} x^2y^2z\,dzdydx$$

$$M_{xy} = \int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} x^2yz^2\,dzdydx$$

Probability

Determined by probability density function, and the sum of all probabilities must equal 1 = the area (volume) under the density function must be 1.

Suppose that the probability density function in 1 variable is $f(x)$ on some interval [a,b].

Then $\int_a^b f(x)\,dx = 1$

Similarly, in the two-variable case: if a probability density function is $f(x,y)$ defined on some region in the plane, then

$$\int_a^b \int_{f(x)}^{g(x)} f(x,y)dydx = 1$$

A probability density function is given by $f(x,y) = kx^2y^2$ on the region bounded by the intervals [0,1]x[0,2]. Find the constant that makes this a valid probability distribution.

$$\int_0^1 \int_0^2 kx^2y^2dydx = \int_0^1 \frac{k}{3}x^2y^3\Big|_0^2 dx = \int_0^1 \frac{8k}{3}x^2dx = \frac{8k}{9}x^3\Big|_0^1 = \frac{8k}{9} = 1$$

$$k = \frac{9}{8}$$

$$f(x,y) = \frac{9}{8}x^2y^2$$

From here, I can calculate probabilities. Change my limits of integration to represent the region where I can finding the probability.

Find the probability that $x < \frac{1}{2}\ and\ y > 1$

$$p = \int_0^{\frac{1}{2}} \int_1^2 \frac{9}{8}x^2y^2\,dydx$$

To find the mean of the probability distribution, multiply by the variable you want to find the mean of under the integral.

$$\bar{x} = \int_0^1 \int_0^2 \frac{9}{8}(x^2y^2)x \, dydx$$

$$\bar{y} = \int_0^1 \int_0^2 \frac{9}{8}(x^2y^2)y \, dydx$$

The end!!!!

Tomorrow, we just review for the final. So bring whatever questions you have, and we'll stay until they are answered, and that will be it.