6/5/2024

Triple Integrals

Triple integrals for volume:

$$\int_a^b \int_{g(x)}^{f(x)} \int_{k(x,y)}^{h(x,y)} dV = \int_a^b \int_{g(x)}^{f(x)} \int_{k(x,y)}^{h(x,y)} dzdydx = \int_a^b \int_{g(x)}^{f(x)} h(x,y) - k(x,y)dydx$$

$h(x,y)$ is the function defining the upper surface of the volume. $k(x,y)$ is the function defining the lower surface of the volume. $f(x)$ is the upper bound in y, and $g(x)$ is the lower bound in y in the xy-plane. And then a and b are the bounds in x.

$$\int_a^b \int_{g(x)}^{f(x)} \int_{k(x,y)}^{h(x,y)} F(x,y,z)dV$$

If we integrate a function over a volume, we can think of this F as a density function, and integrating over the volume would give us the total mass.

You can change the order of integration: there are six possible orders for rectangular coordinates:

$$dzdydx, dzdxdy, dydzdx, dydxdz, dxdydz, dxdzdy$$

Example.

Evaluate $\int\int\int_V zdV$ over the region under the tetrahedron (plane) $x+y+z=1$, and the coordinate planes (x=0,y=0,z=0 or the first octant).

Intercepts: (0,0,1), (1,0,0), (0,1,0)

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dzdydx$$

Set the surface equal to z and then use that and z=0 (xy-plane) as the bounds in z.

After setting up the z-integral, set the two surfaces to be equal and find the bounds in x and y.

$$z = 0, z = 1 - x - y$$
$$0 = 1 - x - y$$
$$y = 1 - x$$

y is the next variable to integrate, so solve for y.

set y=1-x equal to zero (that's other boundary), and find the limits in x.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dzdydx = \int_0^1 \int_0^{1-x} \frac{1}{2}z^2\Big|_0^{1-x-y} dydx = \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2 dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} 1 - x - y - x + x^2 + xy - y + xy + y^2\, dydx =$$

$$\frac{1}{2}\int_0^1 \int_0^{1-x} 1 - 2x - 2y + x^2 + y^2 + 2xydydx = \frac{1}{2}\int_0^1 \left[y - 2xy - y^2 + x^2y + \frac{1}{3}y^3 + xy^2\right]_0^{1-x} dx =$$

$$\frac{1}{2}\int_0^1 1 - x - 2x(1-x) - (1-x)^2 + x^2(1-x) + \frac{1}{3}(1-x)^3 + x(1-x)^2 dx =$$

$$\frac{1}{2}\int_0^1 1 - x - 2x + 2x^2 - 1 + 2x - x^2 + x^2 - x^3 + \frac{1}{3}(1 - 3x + 3x^2 - x^3) + x - 2x^2 + x^3 dx =$$

$$\frac{1}{2}\int_0^1 \frac{1}{3} - x + x^2 - \frac{1}{3}x^3 dx = \frac{1}{2}\left[\frac{1}{3}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4\right]_0^1 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12}\right] = \frac{1}{24}$$

Example.

Evaluate $\int\int\int_V \sqrt{x^2+z^2}dV$ where the region is bounded by the paraboloid $y = x^2 + z^2$ and the plane y=4.

$$\int_{\square}^{\square}\int_{\square}^{\square}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}dydA$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}dydzdx$$

$$x = r\cos\theta, z = r\sin\theta, y = y, x^2 + z^2 = r^2, \theta = \tan^{-1}\left(\frac{z}{x}\right)$$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 rdy(rdrd\theta)$$

Where does the paraboloid intersect with the plane y=4?

$$x^2 + z^2 = 4$$
$$r^2 = 4 \rightarrow r = 2$$

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2dydrd\theta = \int_0^{2\pi}\int_0^2 4 - r^2drd\theta = \int_0^{2\pi}\left[4r - \frac{1}{3}r^3\right]_0^2 d\theta = \int_0^{2\pi} 8 - \frac{8}{3}\, d\theta =$$

$$\int_0^{2\pi}\frac{16}{3}d\theta = \frac{16}{3}(2\pi) = \frac{32}{3}\pi$$

Alternatively, you can switch the y and z, and then set the whole problem up with traditional cylindrical coordinates.

$$\int\int\int_V \sqrt{x^2+y^2}dV, z = x^2 + y^2, z = 4$$

Example.

$$\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)dzdydx$$

Change to dxdzdy

$$z = 0, z = y$$
$$y = 0, y = x^2$$
$$x = 0, x = 1$$

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)dxdzdy$$

$$x = \sqrt{y}$$

Evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)dzdydx$$

Convert to cylindrical coordinates and integrate.

For the function, we can convert coordinates algebraically. $x^2 + y^2 = r^2$. Can convert the z-coordinates algebraically because z=z.

$$\int_0^{2\pi} \int_0^2 \int_r^2 r^2 dzrdrd\theta$$

For the xy-plane, convert geometrically.

$$y = \sqrt{4-x^2}, y = -\sqrt{4-x^2} \rightarrow x^2 + y^2 = 4$$

Circle of radius 2, and the whole circle

$$\int_0^{2\pi} \int_0^2 \int_r^2 r^2 dz{\color{red}r}drd\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dzdrd\theta = \int_0^{2\pi} \int_0^2 r^3(2-r)drd\theta = \int_0^{2\pi} \int_0^2 2r^3 - r^4 drd\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{4}r^4 - \frac{1}{5}r^5\right]_0^2 d\theta = \int_0^{2\pi} 8 - \frac{32}{5}\, d\theta = \frac{8}{5}(2\pi) = \frac{16\pi}{5}$$

Spherical

$$x = \rho\cos\theta\sin\phi, y = \rho\sin\theta\sin\phi, z = \rho\cos\phi, r = \rho\sin\phi, x^2+y^2+z^2 = \rho^2$$

$$\theta = \theta, \theta = \tan^{-1}\left(\frac{y}{x}\right), \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Positive z-axis is $\phi = 0$, xy-plane is $\phi = \frac{\pi}{2}$, negative z-axis is $\phi = \pi$.

In Cylindrical, $dV = rdzdrd\theta$, but in spherical, $dV = \rho^2 \sin\phi \, d\rho d\theta d\phi$

Example. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$

$$\rho\cos\phi = \rho\sin\phi$$
$$\cos\phi = \sin\phi$$
$$\phi = \frac{\pi}{4}$$

$$\rho^2 = \rho\cos\phi$$
$$\rho = \cos\phi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho d\phi d\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho d\phi d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\frac{1}{3}\rho^3\Big|_0^{\cos\phi}\sin\phi\,d\phi d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\cos^3\phi\sin\phi\,d\phi d\theta=$$

$$-\frac{1}{3}\int_0^{2\pi}\frac{\cos^4\phi}{4}\Big|_0^{\frac{\pi}{4}}d\theta=-\frac{1}{3}(2\pi)\left(\frac{1}{4}\right)\left[\left(\frac{1}{\sqrt{2}}\right)^4-1^4\right]=-\frac{1}{3}\left(-\frac{3}{4}\right)\left(\frac{1}{4}\right)2\pi=\frac{\pi}{8}$$

Example.
Convert the integral to spherical then integrate.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dzdydx$$

$$z=\pm\sqrt{1-x^2-y^2}$$
$$z^2=1-x^2-y^2$$
$$x^2+y^2+z^2=1$$

Sphere of radius 1

$$\rho=1$$

$$y=\pm\sqrt{1-x^2}$$
$$y^2=1-x^2$$
$$x^2+y^2=1$$

The bounds in the xy-plane is where the sphere intersects z=0

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1}e^{\rho^3}\rho^2\sin\phi\,d\rho d\theta d\phi$$

$$e^{(x^2+y^2+z^2)^{\frac{3}{2}}}=e^{(\rho^2)^{\frac{3}{2}}}=e^{\rho^3}$$

$$\int_0^{\pi}\int_0^{2\pi}\left[\frac{1}{3}e^{\rho^3}\right]_0^1\sin\phi d\theta d\phi=\frac{1}{3}(e-1)\int_0^{\pi}\int_0^{2\pi}\sin\phi\,d\theta d\phi=\frac{2\pi}{3}(e-1)\int_0^{\pi}\sin\phi\,d\phi=$$

$$-\frac{2\pi}{3}(e-1)\cos\phi|_0^{\pi}=-\frac{2\pi}{3}(e-1)(-1-1)=\frac{4\pi(e-1)}{3}$$

Example. Change to spherical coordinates and integrate.

$$\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}xydzdydx$$

$$\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square}(\rho\cos\theta\sin\phi\,\rho\sin\theta\sin\phi)\rho^2\sin\phi\,d\rho d\theta d\phi = \int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square}\cos\theta\sin\theta\,\rho^4\sin^3\phi\,d\rho d\theta d\phi$$

$$z = \sqrt{x^2+y^2}$$
$$z = r$$
$$\rho\cos\phi = \rho\sin\phi$$
$$\phi = \frac{\pi}{4}$$

$$z = \sqrt{2-x^2-y^2}$$
$$z^2 = 2-x^2-y^2$$
$$x^2+y^2+z^2 = 2$$
$$\rho^2 = 2$$
$$\rho = \sqrt{2}$$

$$x^2+y^2 = 2-x^2-y^2$$
$$2x^2+2y^2 = 2$$
$$x^2+y^2 = 1$$

Circle of radius 1 where the cone and the sphere meet.
$y = \sqrt{1-x^2}, y = 0$ is only the top half of the circle, and moreover, x in 0 to 1, is only positive, so it’s actually the first quadrant of the circle.

So theta will be between 0 and pi/2

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{2}}\cos\theta\sin\theta\,\rho^4\sin^3\phi\,d\rho d\theta d\phi = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\frac{\rho^5}{5}\bigg|_0^{\sqrt{2}}\cos\theta\sin\theta\sin^3\phi\,d\theta d\phi =$$

$$\frac{4\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\sin^3\phi\,d\theta d\phi = \frac{4\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}\frac{1}{2}\sin^2\theta\bigg|_0^{\frac{\pi}{2}}\sin^3\phi\,d\phi = \frac{2\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}\sin^3\phi\,d\phi =$$

$$\frac{2\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}\sin^2\phi\,(\sin\phi)d\phi = \frac{2\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}(1-\cos^2\phi)(\sin\phi)d\phi$$

$$u = \cos\phi,\ du = -\sin\phi$$
$$\int(1-u^2)(-du) = -u+\frac{1}{3}u^3$$

$$\frac{2\sqrt{2}}{5}\int_0^{\frac{\pi}{4}}(1-\cos^2\phi)(\sin\phi)d\phi = \frac{2\sqrt{2}}{5}\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\frac{\pi}{4}} =$$

$$\frac{2\sqrt{2}}{5}\left[-\frac{1}{\sqrt{2}}+\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3-\left(-1+\frac{1}{3}(1)\right)\right] = \frac{2\sqrt{2}}{5}\left[\frac{1}{6\sqrt{2}}-\frac{1}{\sqrt{2}}+1-\frac{1}{3}\right] =$$

$$\frac{2\sqrt{2}}{5}\left[\frac{2}{3}-\frac{5}{6\sqrt{2}}\right]$$

This is the end of the material for Exam #2 (on Monday).

(line integrals to triple integrals)