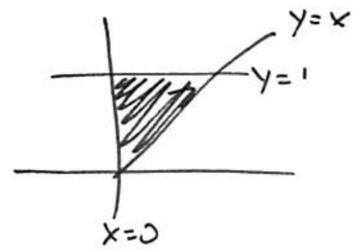
MTH 265, Quiz #10, Summer 2024 Name KEY

Instructions: Show all work. Use exact answers unless otherwise asked to round.

1. Evaluate the integral $\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$ by reversing the order of integration.

$$\int_0^1 \int_0^y e^{x/y}\,dx\,dy = \int_0^1 y e^{x/y}\Big|_0^y dy$$

$$\int_0^1 y e^1\,dy = \frac{e}{2} y^2\Big|_0^1 = \frac{e}{2}$$



2. Set up and evaluate $\iiint_Q x\,dV$ where Q is bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x, z = 0$ in the first octant. Use an appropriate coordinate system.

equivalent to $\iiint_Q z\,dV$ $\quad y^2+x^2=9,\ z=0,\ y=3z,\ x=0$ first octant

$r=3 \qquad \frac{1}{3}r\cos\theta = z$

$$\int_0^{\pi/2}\int_0^3\int_0^{\frac{1}{3}r\cos\theta} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 \frac{1}{2}z^2\Big|_0^{\frac{1}{3}r\cos\theta} r\,dr\,d\theta =$$

$$\frac{1}{18}\int_0^{\pi/2}\int_0^3 r^3\cos^2\theta\,dr\,d\theta = \frac{1}{18}\cdot\frac{1}{4}r^4\Big|_0^3 \int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta =$$

$$\frac{81}{144}\left[\theta + \frac{1}{2}\sin 2\theta\right|_0^{\pi/2} = \frac{9}{16}\left[\pi/2 + 0 - 0 - 0\right] = \frac{9\pi}{32}$$

3. Set up and evaluate $\iiint_Q x e^{x^2+y^2+z^2}\,dV$ where Q is the portion of the unit ball $x^2 + y^2 + z^2 \le 1$ that lies in the first octant. Use an appropriate coordinate system.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\cos\theta\sin\varphi\, e^{\rho^2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3 e^{\rho^2}\cos\theta\sin^2\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\frac{(\rho^2-1)}{2}e^{\rho^2}\Big|_0^1\cos\theta\sin^2\varphi\,d\theta\,d\varphi =$$

$$\frac{1}{2}\int_0^{\pi/2}\sin^2\varphi\,d\varphi\cdot\underbrace{\sin\theta\Big|_0^{\pi/2}}_{=1} = \frac{1}{4}\int_0^{\pi/2}1-\cos 2\varphi\,d\varphi = \frac{1}{4}\left[\varphi - \frac{1}{2}\sin 2\varphi\right|_0^{\pi/2} =$$

$$\frac{1}{4}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{8}$$

4. Consider the space curve $\vec{r}(t) = t\hat{\imath} + e^t\hat{\jmath} + e^{-t}\hat{k}$.

   a. Find $\vec{r}'(t)$

$$r'(t) = 1\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}$$

   b. Find $\|\vec{r}'(t)\|$.

$$\|r'(t)\| = \sqrt{1 + e^{2t} + e^{-2t}}$$

   c. Are there any points at which $\|\vec{r}'(t)\|$ reaches an extremum? (minimum or maximum?)

minimum when $t = 0$ $\rightarrow$ $\|r'(t)\| = \sqrt{3}$

no maximum $\Rightarrow \infty$ as $t \rightarrow \infty$

   d. Find the unit tangent vector $\vec{T}(t)$.

$$T(t) = \frac{1\hat{\imath} + e^t\hat{\jmath} - e^{-t}\hat{k}}{\sqrt{1 + e^{2t} + e^{-2t}}}$$

5. Find the unit normal vector of $\vec{r}(t) = \cos 4t\,\hat{\imath} + t\hat{\jmath} - \sin 4t\,\hat{k}$.

$$r'(t) = -4\sin 4t\,\hat{\imath} + 1\hat{\jmath} - 4\cos 4t\,\hat{k} \qquad \|r'(t)\| = \sqrt{16\cos^2 4t + 1 + 16\sin^2 4t} = \sqrt{17}$$

$$T(t) = \frac{-4\sin 4t\,\hat{\imath} + 1\hat{\jmath} - 4\cos 4t\,\hat{k}}{\sqrt{17}}$$

$$T'(t) = \frac{1}{\sqrt{17}}(-16\cos t\,\hat{\jmath} + 0\hat{\jmath} + 16\sin 4t\,\hat{k}) = \frac{16}{\sqrt{17}}(-\cos 4t\,\hat{\imath} + \sin 4t\,\hat{k})$$

$$\|T'(t)\| = \frac{16}{\sqrt{17}} \qquad N(t) = -\cos 4t\,\hat{\imath} + \sin 4t\,\hat{k}$$

6. Find the directional derivative for the function $f(x, y) = x^2y - e^{x-y}$ at the point $(1,1)$ in the direction of $\langle 2, -5\rangle$. $= \vec{u}$ $\quad \hat{u} = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle$

$\|\hat{u}\| = \sqrt{4 + 25} = \sqrt{29}$

$$\nabla f = \langle 2xy - e^{x-y}, x^2 + e^{x-y} \rangle \qquad \nabla f(1,1) = \langle 1, 2 \rangle$$

$$\nabla f \cdot \hat{u} = \langle 1, 2 \rangle \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle = \frac{2}{\sqrt{29}} - \frac{10}{\sqrt{29}} = \frac{-8}{\sqrt{29}}$$