MTH 265, Quiz #12, Summer 2024 Name KEY

Instructions: Show all work. Use exact answers unless otherwise asked to round.

1. Consider the function $x = \sqrt{y^2 + z^2}$. Identify the surface. Convert the surface to parametric surface form $\vec{r}(u, v)$. Find the equation of the tangent plane at $(5,3,4)$.

Cone wrapped around x-axis

$r = x = u$

$r(u,v) = u\hat{i} + u\cos v\hat{j} + u\sin v\hat{k}$

$u = 5,\ v = \tan^{-1}\left(\frac{4}{3}\right)$

$\cos v = \frac{3}{5},\ \sin v = \frac{4}{5}$

$\vec{r}_u = 1\hat{i} + \cos v\hat{j} + \sin v\hat{k}$

$\vec{r}_v = 0\hat{i} - u\sin v\hat{j} + u\cos v\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = (u\cos^2 v + u\sin^2 v)\hat{i} - (u\cos v - 0)\hat{j} + (-u\sin v - 0)\hat{k}$$

$$= u\hat{i} - u\cos v\hat{j} - u\sin v\hat{k} = 5\hat{i} - 3\hat{j} - 4\hat{k}$$

$5(x-5) - 3(y-3) - 4(z-4) = 0$

2. Find the arc length of the function $\vec{r}(t) = t^2\hat{\imath} + \ln t\,\hat{\jmath} + t\ln t\,\hat{k}$ on the interval $[1, e]$. After setting up the integral, you may evaluate it numerically (in a calculator).

$r'(t) = 2t\hat{i} + \frac{1}{t}\hat{j} + (\ln t + 1)\hat{k}$

$\|r'(t)\| = \sqrt{4t^2 + \frac{1}{t^2} + (\ln t + 1)^2}$

$S = \int_1^e \sqrt{4t^2 + \frac{1}{t^2} + (\ln t + 1)^2}\, dt \approx 7.043257\ldots$

3. Find the curvature of the function $\vec{r}(t) = t^2\hat{\imath} + \ln t\,\hat{\jmath} + t\ln t\,\hat{k}$ at the point $(e^2, 1, e)$. Then use that to find the radius of curvature.

$r'(t) = 2t\hat{i} + \frac{1}{t}\hat{j} + (\ln t + 1)\hat{k}$

$r''(t) = 2\hat{i} - \frac{1}{t^2}\hat{j} + \frac{1}{t}\hat{k}$

$t = e$

$K(e) = \dfrac{\sqrt{4e^4 + 16e^2 + 9}}{(4e^3 + 4e^2 + 1)^{3/2}} \approx 0.043272$

$R(e) = 1/K \approx 23.1098$

$$r' \times r'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & 1/t & \ln t + 1 \\ 2 & -\frac{1}{t^2} & \frac{1}{t} \end{vmatrix} = \left(\frac{1}{t^2} + \frac{\ln t + 1}{t^2}\right)\hat{i} - (\cancel{2} - 2\ln t - \cancel{2})\hat{j} + \left(-\frac{2}{t} - \frac{2}{t}\right)\hat{k} =$$

$$\left(\frac{\ln t + 2}{t^2}\right)\hat{i} + 2\ln t\hat{j} - \frac{4}{t}\hat{k}$$

$K = \dfrac{\sqrt{\frac{(\ln t + 2)^2}{t^4} + 4\ln^2 t + \frac{16}{t^2}}}{\left[\text{[illegible]}\right]^3}$

4. Find the surface area of the function $z = xy$ over the region bounded inside the cylinder $x^2 + y^2 = 2$.

$F = xy - z \qquad \nabla F = \langle y, x, -1 \rangle \qquad \iint_R \underbrace{\sqrt{y^2+x^2+1}}_{\|\nabla F\|}\, dA$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2+1}\; r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{3}(r^2+1)^{3/2}\Big|_0^{\sqrt{2}} d\theta$$

$$\int_0^{2\pi} \frac{1}{3}[3^{3/2}-1]\,d\theta = \frac{2\pi}{3}[3^{3/2}-1]$$

$u = r^2+1$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

$\frac{1}{2}\int u^{1/2}du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}$

5. Set up the integral needed to find the surface area of the function $\vec{r}(u,v) = u^2\cos v\,\hat{\imath} + u^2\sin v\,\hat{\jmath} + uv\hat{k}$ over the region $0 \le u \le 3, 0 \le v \le 2\pi$. You do not need to integrate.

$\vec{r}_u = 2u\cos v\,\hat{\imath} + 2u\sin v\,\hat{\jmath} + v\hat{k}$

$\vec{r}_v = -u^2\sin v\,\hat{\imath} + u^2\cos v\,\hat{\jmath} + u\hat{k}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2u\cos v & 2u\sin v & v \\ -u^2\sin v & u^2\cos v & u \end{vmatrix} = (2u^2\sin v - u^2v\cos v)\hat{\imath} - (2u^2\cos v + u^2v\sin v)\hat{\jmath} + (2u^3\cos^2 v + 2u^3\sin^2 v)\hat{k}$$

$$= (2u^2\sin v - u^2v\cos v)\hat{\imath} - (2u^2\cos v + u^2v\sin v)\hat{\jmath} + 2u^3\hat{k}$$

$$S = \iint_R \|\vec{r}_u \times \vec{r}_v\|\,dA = \int_0^{2\pi}\int_0^3 u\sqrt{2(\sin v - v\cos v)^2 + (-2\cos v - v\sin v)^2 + 4u^2}\;du\,dv$$