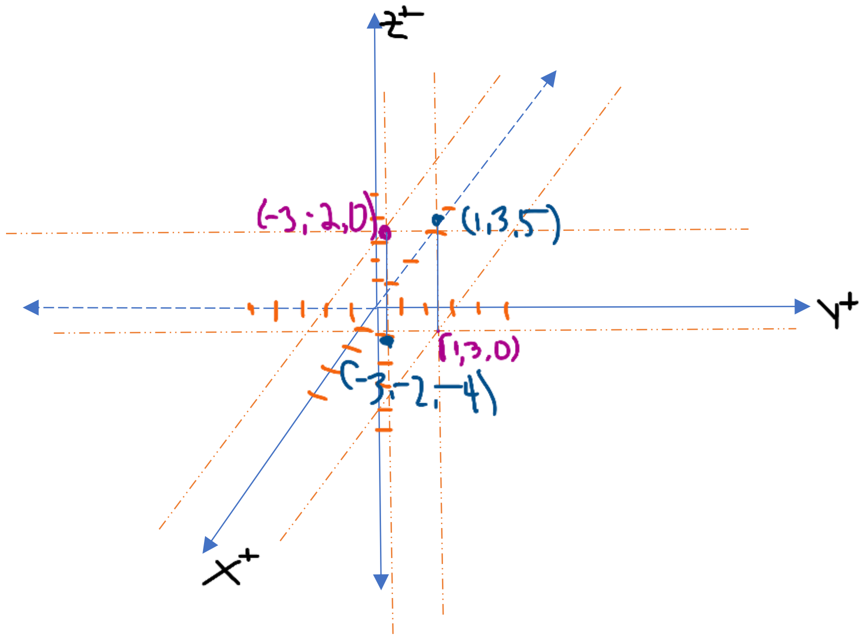
5/19/2025

Introduction to the course
Vectors – in 2D and 3D

Orienting Space: right-hand rule vs. left-hand rule

Index finger is x, your middle finger is y, and thumb is z



Plot the point $(1,3,5)$ (when all the coordinates are positive, this is in the first octant)
Plot the point $(-3,-2,-4)$

Distance formula in 3D:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Midpoint formula

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Recall the equation of a circle:

$$x^2 + y^2 = r^2$$

$$(x - h)^2 + (y - k)^2 = r^2$$

In 3D we have a sphere:

$$x^2 + y^2 + z^2 = \rho^2$$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = \rho^2$$

If our equation has only two variables in it, then the third variable is "free" to be anything… what we get is called a cylinder.

A circle in 2D, in 3D space is a right circular cylinder, because the z-value is free…

https://www.desmos.com/3d
3D surface grapher

Vectors
Plot vectors in 3D, the same way we plot points in 3D… plot the endpoint (the coordinates of the vector) and then connect the endpoint to the origin.

Find the vector between the two points: $P(4,1,5), Q(-2,3,1)$
$\overrightarrow{PQ} = \langle -2-4, 3-1, 1-5\rangle = \langle -6,2,-4\rangle$
$$\overrightarrow{QP} = \langle 6,-2,4\rangle$$

If you are trying to determine if a triangle has a particular angle between two sides (in 12.3)

When you add vectors, you add component by component.
The parallelogram rule still applies (now in 3D)

The magnitude of a vector:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Magnitude and direction.
In two dimensions, we can talk about the angle that a vector makes with the positive x-axis as the direction. But, in 3D we can't do that. We would need two angles. In 3D, we usually use a unit vector instead.

$$\|\overrightarrow{PQ}\| = \sqrt{(-6)^2 + (2)^2 + (-4)^2} = \sqrt{36+4+16} = \sqrt{56} = 2\sqrt{14}$$

Unit vector in the direction of $\overrightarrow{PQ}$

$$\widehat{PQ} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|}$$

$$\widehat{PQ} = \langle -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\rangle$$

$$\hat{\imath} = \langle 1,0,0\rangle, \hat{\jmath} = \langle 0,1,0\rangle, \hat{k} = \langle 0,0,1\rangle$$

$$\overrightarrow{PQ} = -6\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$$

Scalar multiples of vectors:

$$2\vec{v} = \langle 2v_1, 2v_2, 2v_3\rangle$$

Adding vectors:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$$

Dot Product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Also called the scalar product because it converts vectors into a number (scalar).
Also called an inner product

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$$

Where θ is the angle between the vectors.

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

If the dot product is positive, then the cosine is positive, and therefore the angle between the vectors is acute (an angle in first quadrant).
If the dot product is negative, then the cosine is negative, and therefore the angle between the vectors is obtuse (and angle in the second quadrant).

If the dot product is 0, then the angle between the vectors is 90-degrees, and therefore the vectors are orthogonal (perpendicular), i.e. it's a right angle.

Example.
Find the angle between the vectors $\vec{u} = \langle 2, -4, 5 \rangle, \vec{v} = \langle 1, 3, -1 \rangle$

$$\vec{u} \cdot \vec{v} = 2(1) - 4(3) + 5(-1) = 2 - 12 - 5 = -15$$
$$\|\vec{u}\| = \sqrt{4 + 16 + 25} = \sqrt{45}$$
$$\|\vec{v}\| = \sqrt{1 + 9 + 1} = \sqrt{11}$$

$$\cos(\theta) = \frac{-15}{\sqrt{45}\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{-15}{\sqrt{45}\sqrt{11}}\right)$$
$$\theta \approx 2.3106 \ldots radians$$
$$\theta \approx 132.4°$$

Direction angles/Direction cosines

$$\alpha, \beta, \gamma$$

The angle between the unit vector and the vector. So since there are three unit vectors, there are three direction angles (direction cosines).

$$\alpha = \cos^{-1}\left(\frac{u_1}{\|\vec{u}\|}\right)$$

$$\cos(\alpha) = \left(\frac{u_1}{\|\vec{u}\|}\right)$$

$$\beta = \cos^{-1}\left(\frac{u_2}{\|\vec{u}\|}\right)$$

$$\cos(\beta) = \left(\frac{u_2}{\|\vec{u}\|}\right)$$

$$\gamma = \cos^{-1}\left(\frac{u_3}{\|\vec{u}\|}\right)$$

$$\cos(\gamma) = \left(\frac{u_3}{\|\vec{u}\|}\right)$$

Work:

$$W = \vec{F} \cdot \vec{d}$$