5/20/2025

Cross Product
Vector-Valued Functions
Vector Fields

Cross Product, also called a vector product, or an outer product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - v_2 u_3)\hat{\imath} - (u_1 v_3 - v_1 u_3)\hat{\jmath} + (u_1 v_2 - v_1 u_2)\hat{k}$$

The result is a vector orthogonal (perpendicular) to the original two vectors.

This only works in three dimensions.

Find the cross product between $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 1, -4, -2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 2 \\ 1 & -4 & -2 \end{vmatrix} = \left(\begin{vmatrix} -1 & 2 \\ -4 & -2 \end{vmatrix}\right)\hat{\imath} - \left(\begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\right)\hat{\jmath} + \left(\begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix}\right)\hat{k} =$$

$$\big((-1)(-2) - (-4)(2)\big)\hat{\imath} - \big((3)(-2) - (1)(2)\big)\hat{\jmath} + \big((3)(-4) - (1)(-1)\big)\hat{k} =$$

$$\langle (2+8), -(-6-2), (-12+1) \rangle = \langle 10, 8, -11 \rangle = \vec{w}$$

$$\vec{u} \cdot \vec{w} = 3(10) - 1(8) + 2(-11) = 30 - 8 - 22 = 0$$

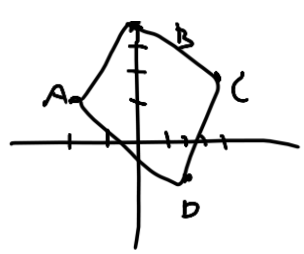
$$\vec{v} \cdot \vec{w} = 1(10) - 4(8) - 2(-11) = 10 - 32 + 22 = 0$$

One interpretation of the cross product (the magnitude of the cross product) is its relationship to the angle between two vectors:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

If you have a parallelogram defined from one vertex by two vectors, then the magnitude of the cross product is equal to the area of the parallelogram.

Example. Find the area of the parallelogram defined by A(-2,1), B(0,4), C(4,2), D(2,-1)



$$\overrightarrow{AD} = \langle 4, -2, 0\rangle, \overrightarrow{AB} = \langle 2,3,0\rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{vmatrix} = \langle (0-0), -(0-0), (-4-12)\rangle = \langle 0,0,-16\rangle$$

$$\left\|\overrightarrow{AB} \times \overrightarrow{AD}\right\| = 16$$

Properties of the cross product:
If you switch the order of the vectors in the cross product, the resulting vector is the negative of the one in the original order.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Triple Scalar Product

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

Its geometric use is for finding the volume of a parallelepiped (slanty box)

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Find the volume of the parallelepiped with the given adjacent edges PQ, PR, PS:
P(-2,1,0), Q(2,3,2), R(1,4,-1),S(3,6,1)

$$\overrightarrow{PQ} = \langle 4,2,2\rangle, \overrightarrow{PR} = \langle 3,3,-1\rangle, \overrightarrow{PS} = \langle 5,5,1\rangle$$

$$\overrightarrow{PQ} \cdot \left(\overrightarrow{PR} \times \overrightarrow{PS}\right) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = (3+5)4 - (3+5)2 + (15-15)2 = 32 - 16 + 0 = 16$$

Torque is a cross product

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Vector-valued Functions and Curves in Space

In the past, when we wanted to represent curves that were not functions in the plane, we could use parametric equations (a set of them) to represent the curve. In three dimensions, a single equation can only produce a surface, it can't produce a curve. To produce a curve in space, we need to use parametric equations (or we can define the curve as the intersection of surfaces).

$$x = x(t)$$
$$y = y(t)$$
$$z = z(t)$$

A vector-valued function is essentially a vector whose components are the set of parametric functions for a curve.

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

In two dimensions:

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

The parametrization for a circle is $x = r\cos t\,, y = r\sin t$

$$\vec{r}(t) = \langle 3\cos t\,, 3\sin t \rangle$$

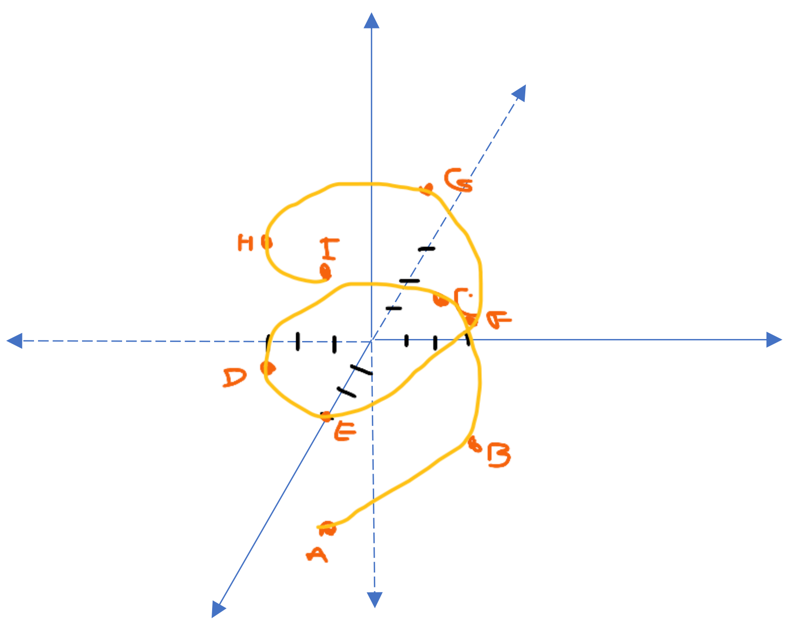
This is a circle of radius 3 in the x-y plane.

$$\vec{r}(t) = \langle 3\cos t\,, 3\sin t\,, 2 \rangle$$

This is a circle of radius 3, at a height of 2 in the z-direction.

Helix.

$$\vec{r}(t) = \langle 3\cos t\,, 3\sin t\,, \frac{1}{2}t \rangle$$



t	Letter	$x = 3\cos t$	$y = 3\sin t$	$z = \frac{1}{2}t$
-2π	A	3	0	$-\pi$

$-\frac{3\pi}{2}$	B	0	3	$-\frac{3\pi}{4}$
$-\pi$	C	-3	0	$-\frac{\pi}{2}$
$-\frac{\pi}{2}$	D	0	-3	$-\frac{\pi}{4}$
0	E	3	0	0
$\frac{\pi}{2}$	F	0	3	$\frac{\pi}{4}$
π	G	-3	0	$\frac{\pi}{2}$
$\frac{3\pi}{2}$	H	0	-3	$\frac{3\pi}{4}$
2π	I	3	0	$\frac{\pi}{2}$

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, kt\rangle$$

Wraps around the z-axis

$$\vec{r}(t) = \langle 3\cos t, kt, 3\sin t\rangle$$

Wraps around the y-axis

$$\vec{r}(t) = \langle kt, 3\cos t, 3\sin t\rangle$$

Wraps around the x-axis

Equations of lines in three-space

1) Intersect two planes
2) Direction vector and a point

Find the vector-valued function for the equation of the line connecting the points $(-2,1,4)$, $(5,3,0)$

$$\vec{u} = \langle 7,2,-4\rangle$$

Direction vector

$$\vec{r}(t) = \langle at + x_0, bt + y_0, ct + z_0\rangle$$

Where $\langle a, b, c\rangle$ is the direction vector and (x_0, y_0, z_0) is a point on the line

$$\vec{r}(t) = (x_0, y_0, z_0) + \langle a, b, c\rangle t$$

$$\vec{r}(t) = \langle 7t - 2, 2t + 1, -4t + 4\rangle$$

When t=0, you are on the first point (-2,1,4), and when t=1, you are on the second point (5,3,0).

Symmetric Equations for a line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are derived from the vector-valued function form, but we solve for t in each component, and then set all the t's equal to each other.
You can't use this if any of your vector components are 0.

Suppose c=0,

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}, z=z_0$$

Intersection of surfaces:
Suppose we want to find the equation of the line where the planes $3x+4y-z=6$ intersects with the plane $2x-4y+2z=7$

$$z=3x+4y-6$$
$$z=-x+2y+\frac{7}{2}$$

$$3x+4y-6=-x+2y+\frac{7}{2}$$

$$4x+2y=\frac{19}{2}$$

$$2x+y=\frac{19}{4}$$

$$y=-2x+\frac{19}{4}$$

$$x=t, y=-2t+\frac{19}{4}$$

$$z=3x+4y-6=3t+4\left(-2t+\frac{19}{4}\right)-6=3t-8t+19-6=-5t+13$$

$$\vec{r}(t)=\langle t,-2t+\frac{19}{4},-5t+13\rangle$$

Limits and Domain ~~and Range~~ of vector-valued functions

Limits of vector-valued functions apply component by component

$$\lim_{t\to 0}\langle\frac{1}{t^2+1}, \ln(t+1), e^t\rangle=\langle\lim_{t\to 0}\frac{1}{t^2+1}, \lim_{t\to 0}\ln(t+1), \lim_{t\to 0}e^t\rangle=\langle 1,0,1\rangle$$

Domain of a vector-valued function is going to depend on the interval where all the components are defined simultaneously.

Find the domain of the vector-valued function $\vec{r}(t)=\langle\frac{t^2}{t-1}, \sqrt{t+8}, \ln t\rangle$

Component 1: $t\neq 1$
Component 2: $t+8\geq 0, t\geq -8$
Component 3: $t>0$

Domain: $(0,1) \cup (1, \infty)$

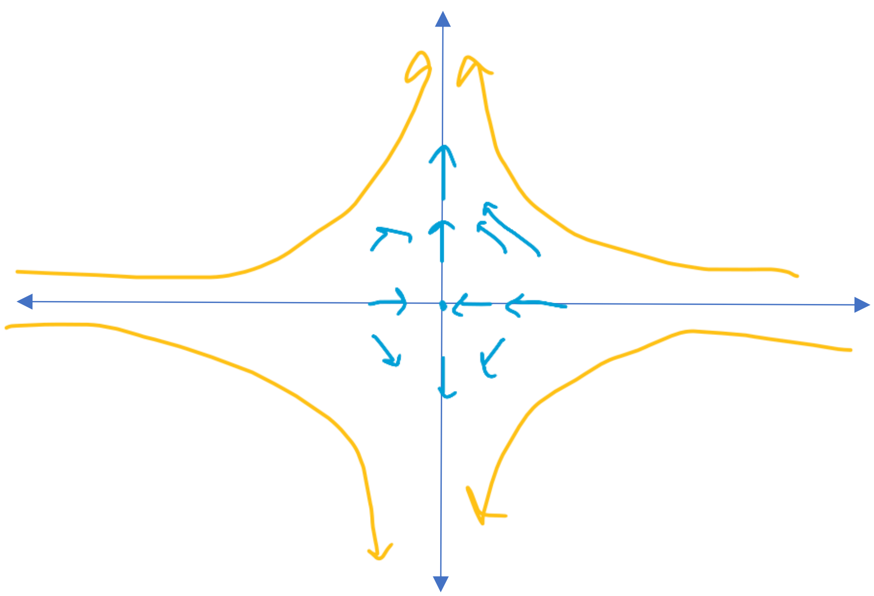
Vector Fields

$$\vec{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$$

Each position in space is associated with a vector

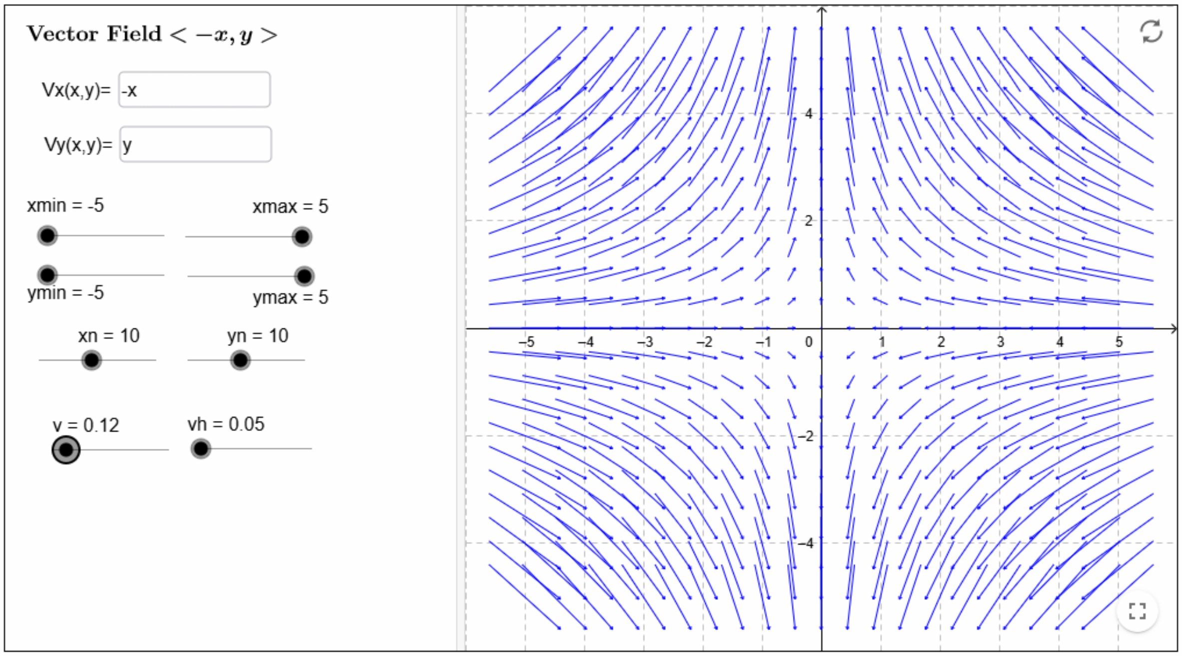
In 2-dimensions:

$$\vec{F}(x, y) = \langle -x, y \rangle$$



x	y	$\langle -x, y \rangle$
0	0	$\langle 0,0 \rangle$
1	0	$\langle -1,0 \rangle$
0	1	$\langle 0,1 \rangle$
-1	0	$\langle 1,0 \rangle$
0	-1	$\langle 0,-1 \rangle$
1	1	$\langle -1,1 \rangle$
1	-1	$\langle -1,-1 \rangle$
-1	-1	$\langle 1,-1 \rangle$
-1	1	$\langle 1,1 \rangle$
2	0	$\langle -2,0 \rangle$
0	2	$\langle 0,2 \rangle$
2	1	$\langle -2,1 \rangle$

https://www.geogebra.org/m/QPE4PaDZ


Vector Field $< -x, y >$
Vx(x,y)= -x
Vy(x,y)= y
xmin = -5
xmax = 5
ymin = -5
ymax = 5
xn = 10
yn = 10
v = 0.12
vh = 0.05
−5
−4
−3
−2
−1
0
1
2
3
4
5
4
2
−2
−4