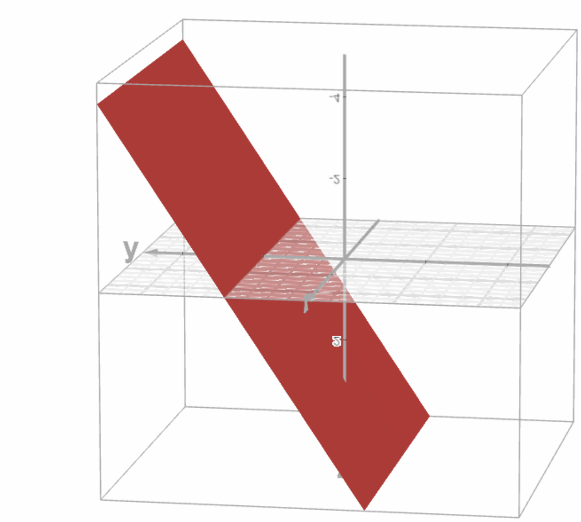
5/21/2025

Equations of Planes
Surfaces in 3D
Quadric Surfaces
Functions of 2 or more variables

3d surface graphers:
https://c3d.libretexts.org/CalcPlot3D/index.html
https://www.geogebra.org/3d?lang=en
https://academo.org/demos/3d-surface-plotter/
https://www.wolframalpha.com/input/?i=3d+plot
https://www.math3d.org/
https://www.desmos.com/3d

Equation of the plane in 3D: any linear equation in 3-space is a plane.
Recall that in 2D, any linear equation is a straight line.

A plane is defined by a vector perpendicular to the plane: normal vector
And a point in the plane.



Equation of the plane for $\vec{n} = \langle a, b, c\rangle$, point in the plane $P(x_0, y_0, z_0)$ the equation of the plane is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

If you multiple this out:

$$ax + by + cz = d$$

Example.
Find the equation of the plane with the normal vector $\langle 3, -1,2\rangle$ passing through the point $(0,4,1)$.

$$3(x - 0) - 1(y - 4) + 2(z - 1) = 0$$

$$3x - y + 4 + 2z - 2 = 0$$
$$3x - y + 2z = -2$$

Other information that can get you to a plane:

1) Give you three points in the plane.
2) Two lines (or two vectors) in the plane (plus one point)

In three points, you would find vectors between the three points (two of them), and find the cross product of the vectors to get the vector perpendicular to the plane (normal vector). Then use one of the points.

Example.
Given the points $P(1,3,2), Q(3,-1,6), R(5,2,0)$. Find the equation of the plane containing these points.

Find two vectors:

$$\overrightarrow{PQ} = \langle 2,-4,4\rangle, \overrightarrow{PR} = \langle 4,-1,-2\rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = (8+4)\hat{\imath} - (-4-16)\hat{\jmath} + (-2+16)\hat{k} = \langle 12,20,14\rangle$$

Use $\langle 12,20,14\rangle$ or rescale to smaller numbers $\langle 6,10,7\rangle$

$$6(x-1) + 10(y-3) + 7(z-2) = 0$$

Distance between point and a plane

$$D = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

Distance between a point and a line

$$D = \frac{\left\|\overrightarrow{PQ} \times \vec{u}\right\|}{\|\vec{u}\|}$$

Find the distance between the point $P(1,2,3)$ and the plane $3x + 2y - z = 10$

P is the point not on the line or the plane.
Q is a point on the line or the plane
$\overrightarrow{PQ}$ find the vector from the two points
Extract the direction vector from the line $\vec{u}$ or the normal vector from the plane $\vec{n}$.
Then plug into the formula.

Q, x=0, y=0…$z = -10$

$$(0,0,-10)$$

$$\overrightarrow{PQ} = \langle -1, -2, -13 \rangle$$
$$D = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\|\vec{n}\|}$$
$$\vec{n} = \langle 3,2,-1 \rangle$$

$$\overrightarrow{PQ} \cdot \vec{n} = -3 - 4 + 13 = 6$$
$$\|\vec{n}\| = \sqrt{9+4+1} = \sqrt{14}$$

$$D = \frac{6}{\sqrt{14}}$$

Example.
Find the distance between the point $P(3,1,-5)$ and the line $\frac{x-1}{4} = \frac{y+2}{2} = \frac{z-3}{-2}$

As a reminder, $\vec{r}(t) = \langle 4t+1, 2t-2, -2t+3 \rangle$
$$\vec{u} = \langle 4,2,-2 \rangle$$

$$Q(1,-2,3)$$

$$\overrightarrow{PQ} = \langle -2,-3,8 \rangle$$

$$D = \frac{\left\|\overrightarrow{PQ} \times \vec{u}\right\|}{\|\vec{u}\|}$$

$$\overrightarrow{PQ} \times \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -3 & 8 \\ 4 & 2 & -2 \end{vmatrix} = \langle (6-16), -(4-32), (-4+12) \rangle = \langle -10,28,8 \rangle$$

$$\left\|\overrightarrow{PQ} \times \vec{u}\right\| = \sqrt{100+784+64} = \sqrt{948} = 2\sqrt{237}$$
$$\|\vec{u}\| = \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$$

$$D = \frac{2\sqrt{237}}{2\sqrt{6}} = \frac{\sqrt{237}}{\sqrt{6}}$$

The angle of intersection of two planes

Find the angle between their two normal vectors.

Find the angle of intersection between the plane $3x + 2y + 4z = 12, 2x - 4y + z = 9$

$$\vec{n}_1 = \langle 3,2,4 \rangle, \vec{n}_2 = \langle 2,-4,1 \rangle$$

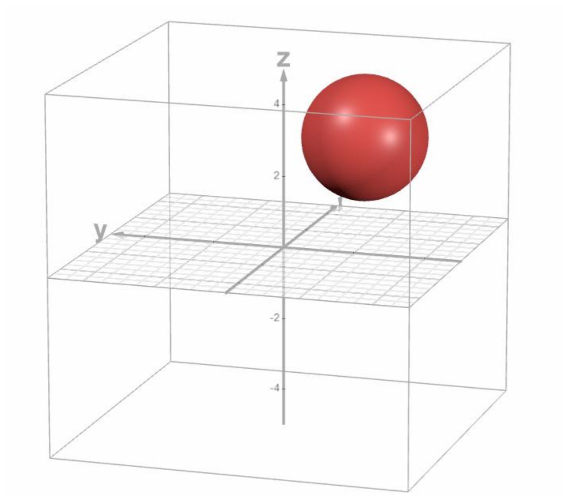
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

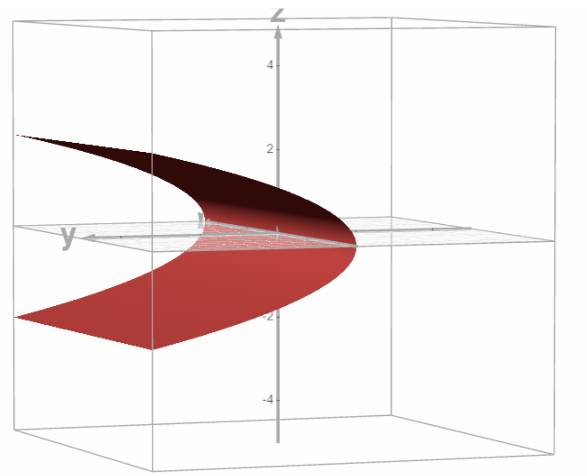
$$\cos\theta = \frac{6-8+4}{\sqrt{9+4+16}\sqrt{4+16+1}} = \frac{2}{\sqrt{29}\sqrt{21}}$$

$$\cos^{-1}\left(\frac{2}{\sqrt{29}\sqrt{21}}\right) \approx 85.3°, 1.48966\ldots radians$$

Surfaces in 3D

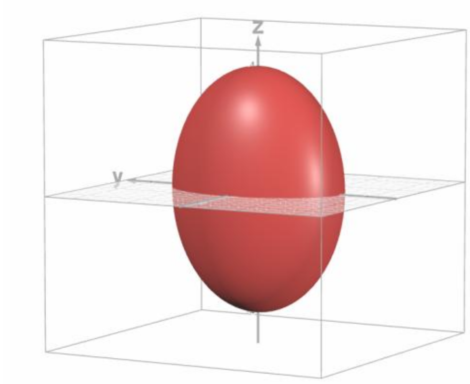
Spheres, cylinders (two-dimensional equations extended to three dimensions)



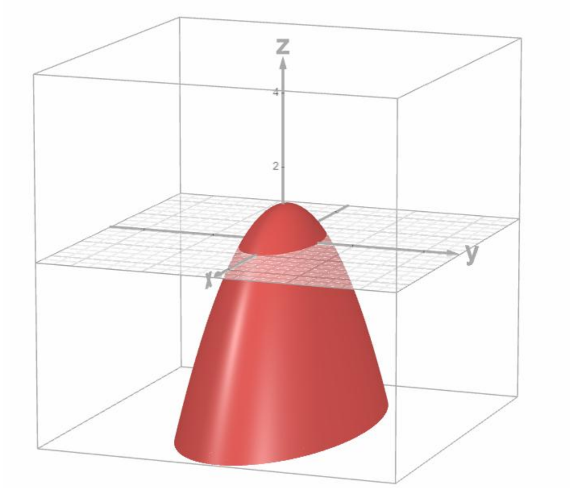


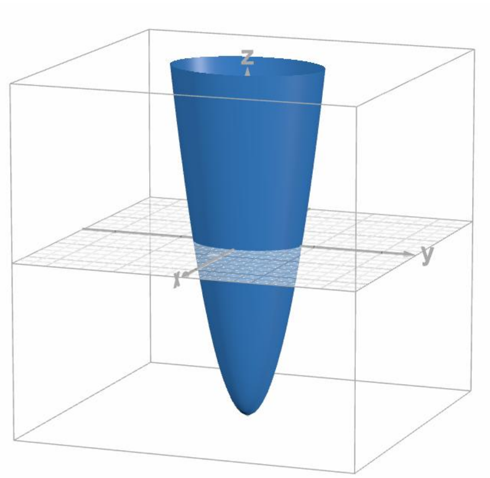
Quadric Surfaces:
Equations in three variables that have powers no larger than 2

Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



Elliptic Paraboloid



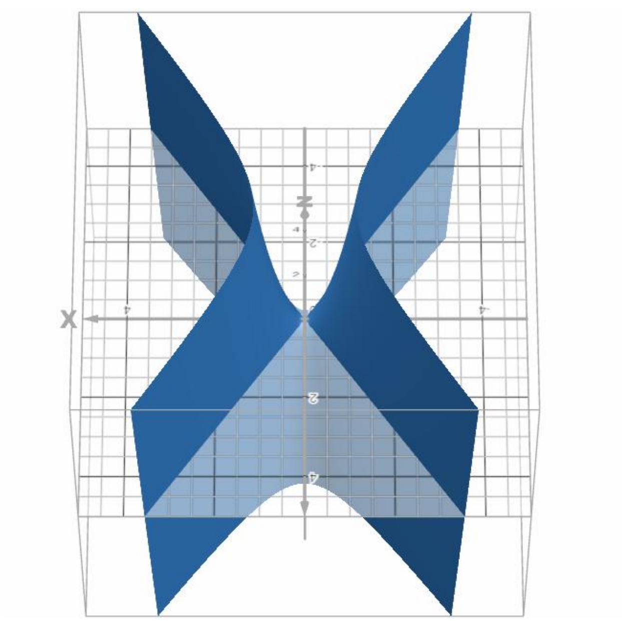


One linear term, two positive squared terms (potentially with difference coefficients) a function.

$$z = 2x^2 + 4y^2$$

Hyperbolic Paraboloid

Also have a linear term, but the signs of the squared terms are different.



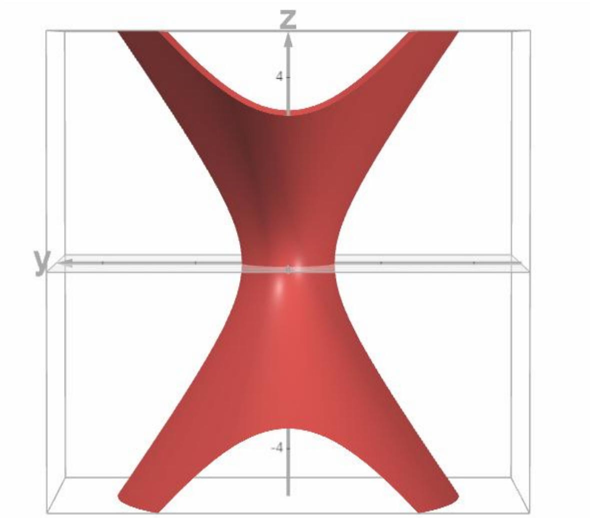
$$z = 2x^2 - 4y^2$$

Hyperboloid of 1 Sheet/the hyperboloid of two sheets/Cone

All variables are squared, and some sign changes: 1 negative, hyperboloid of one sheet, and two negatives makes a hyperboloid of two sheets, and the constant is zero makes the cone.

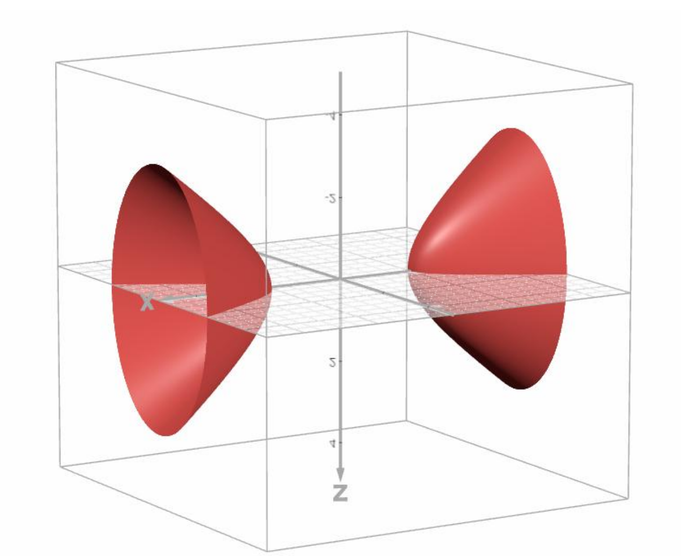
In the hyperboloid of one sheet, the negative sign marks the axis the graph wraps around, so if z is negative, it wraps around the z-axis.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



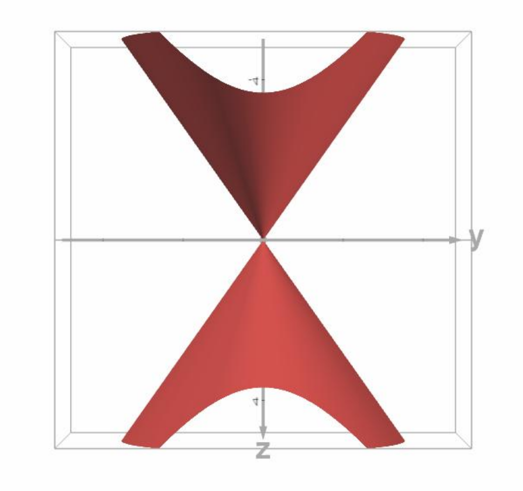
If there are two negatives, we get the hyperboloid of two sheets, and the positive axis is the transverse axis (the one graph wraps around)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



Cone: make the constant zero:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



$$z^2 = x^2 + y^2$$

Functions of two or more variables

$$z = f(x, y)$$
$$w = f(x, y, z)$$

$$g(s, t, u, v)$$

Evaluate a function of several variables.

$$z = f(x, y) = 4\cos x + \ln(xy) - e^{\frac{y}{2}} + y^2$$

Find $f(1,1)$

$$f(1,1) = 4\cos 1 + \ln(1) - e^{\frac{1}{2}} + 1 = 4\cos 1 - e^{\frac{1}{2}} + 1$$

Find the domain and range of a function of 2 or more variables.

$$f(x, y) = \frac{1}{x - 1} + \ln y$$

$$D: \{(x, y) | x \neq 1, y > 0\}$$
$$R: (-\infty, \infty)$$

$$f(x, y) = \sqrt{x^2 + y^2 - 4}$$

$$D: \{(x, y) | x^2 + y^2 \geq 4\}$$
$$R: [0, \infty)$$

We can't write the domain in interval notation anymore.

We can still write the range in interval notation because it is just one variable.

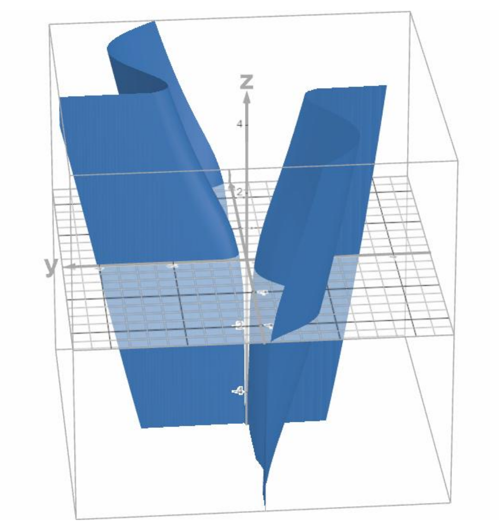
$$f(x,y) = \sqrt{x^2 + y^2 + 4}$$
$$D: \{(x,y) | x, y \text{ in } \mathcal{R}\}$$
$$R: [2, \infty)$$

Find the domain and range for this:

$$f(x,y) = 4\cos x + \ln(xy) - e^{\frac{y}{2}} + y^2$$

$$D: \{(x,y) | xy > 0\}$$

$$R: (-\infty, \infty)$$



Contour curves or level curves
Traces

Trace is essentially a cross section of the graph as projected onto the xz plane or the yz plane.
Set y=0, or x=0
What does the graph look like from one side (as it intersects the plane)

$$z = x^2 + y$$

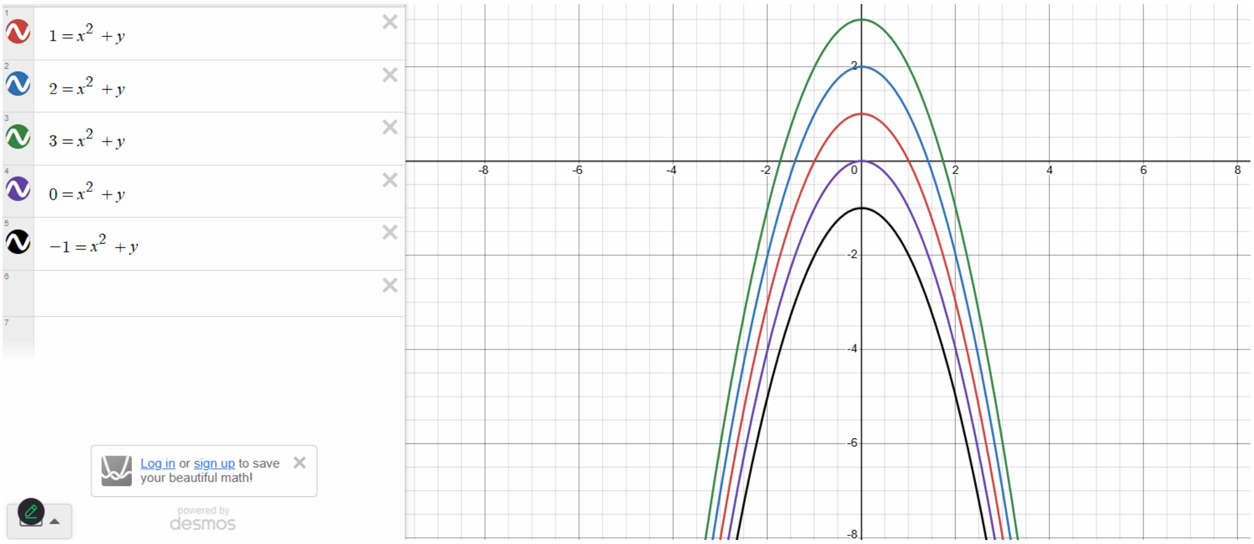
If x=0, the graph looks like y=z, which is a straight line.

If y=0, the graph looks like $z = x^2$

For the contour curves, we don't just do one value of z, we plot the graph at several different values for z.

$$k = x^2 + y$$

https://www.desmos.com/calculator



Next time, more examples of 3D shapes in 2D graphs.
Other coordinate systems