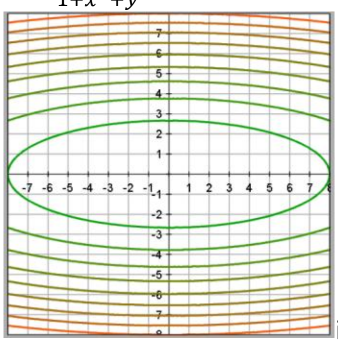
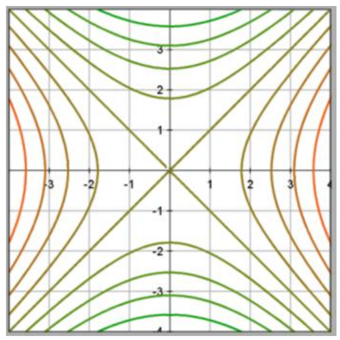
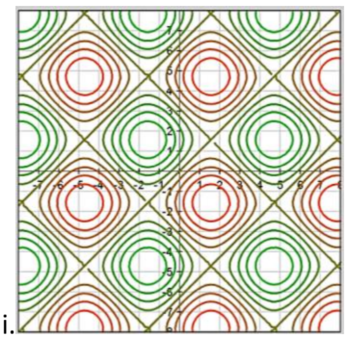
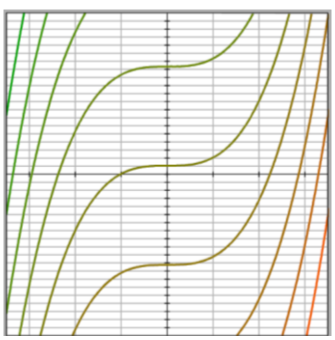
MTH 277, Homework #5, Spring 2025 Name ____

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. Graph at least 5 contour curves of each function.
   a. $z = (y - 2x)^2$
   b. $z = \frac{y}{x^2+y^2}$
   c. $f(x, y) = ye^x$
   d. $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$

2. Match each of the contour curves below to the function that produced them.
   a. $z = x^2 - y^2$
   b. $z = x^3 - y$
   c. $z = \sqrt{y^2 - x^2}$
   d. $z = \frac{x-y}{1+x^2+y^2}$
   e. $z = x^2 + 9y^2$
   f. $z = \ln(x^2 + y^2)$
   g. $z = \sin x - \sin y$
   h. $z = |x| - |y|$

i.

iv.

vii.

ii.

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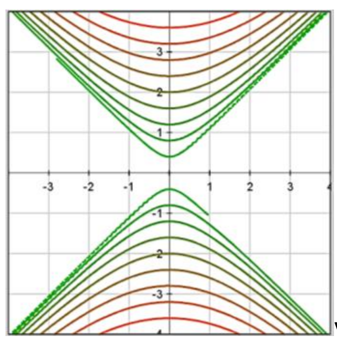
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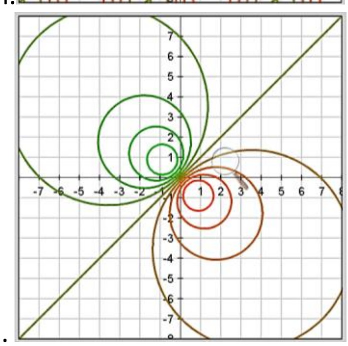
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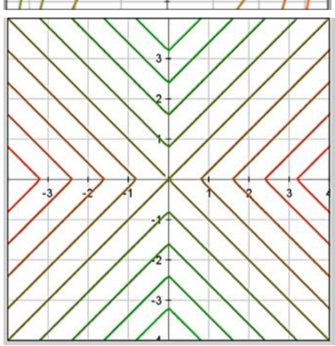
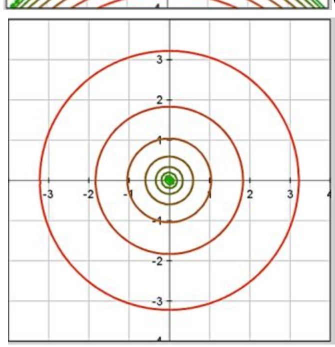

vi.


3. Find $\vec{\nabla} f$ and $\nabla^2 f$ for each function.
   a. $f(x, y) = e^{xy}$
   c. $f(x, y) = x^2 - y^2 - 2x + 6y + 13$

b. $f(x,y,z) = x\ln(y-2z)$

d. $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

4. For each vector field, find $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$.
   a. $\vec{F}(x,y) = (xy-2)\hat{\imath} + (y^2-10)\hat{\jmath}$
   b. $\vec{F}(x,y) = \tan(3x-4y)\hat{\imath} + \ln(1+x^2+2y^2)\hat{\jmath}$
   c. $\vec{F}(x,y,z) = xyz\hat{\imath} + (2x-3z)\hat{\jmath} + (x^2+yz)\hat{k}$
   d. $\vec{F}(x,y,z) = \tan^{-1} yz\,\hat{\imath} + e^{yz}\hat{\jmath} + \sqrt[3]{z+1}\hat{k}$

5. For the function $f(x,y,z) = xy + z^2 - 6$, and the vector fields $\vec{F}(x,y,z) = xy\hat{\imath} - 2y\hat{\jmath} + x^2\sin(z)\hat{k}$ and $\vec{G}(x,y,z) = \ln(x)\,\hat{\imath} + 2e^z\hat{\jmath} - 3y\hat{k}$, calculate the following expressions.
   a. $\nabla\times(\vec{F}\times\vec{G})$
   b. $\nabla\cdot(\vec{F}\times\vec{G})$
   c. $\nabla\cdot(\nabla f)$
   d. $\nabla\times(\nabla\times\vec{F})$
   e. $\nabla\cdot(\mathrm{f}\vec{F})$

6. Find gradient vector field for the scalar function, i.e. find the conservative vector field for the given potential function.
   a. $f(x,y) = \sin 3x \cos 4y$
   b. $f(x,y,z) = x \arcsin yz$
   c. $f(x,y,z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$

7. Determine if the vector field is conservative. If it is, find a potential function for the vector field.
   a. $F(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}(x\vec{i} + y\vec{j})$
   b. $F(x,y) = xe^{x^2y}(2y\vec{i} + x\vec{j})$
   c. $F(x,y,z) = \sin y\vec{i} - x\cos y\vec{j} + \vec{k}$
   d. $F(x,y) = \dfrac{1}{xy}(y\vec{i} - x\vec{j})$
   e. $F(x,y) = 3x^2y^2\vec{i} + 3x^3y\vec{j}$
   f. $F(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^3\vec{k}$

8. Sketch the trace of each function at $x = 0$, and $y = 0$, and three contour curves. Use that information to sketch the surface. Confirm the result with technology.
   a. $x^2 + z^2 = 1$
   b. $z = \sin y$
   c. $x^2 + 4y^2 + 9z^2 = 1$
   d. $z = 1 - y^2$
   e. $x^2 = y^2 + 4z^2$
   f. $-x^2 + y^2 - z^2 = 1$

9. Find the total differential. Evaluate at $f(1,2)$ and use that information to estimate the value of $f(1.05,1.9)$ or evaluate $g(1,2,0)$ to estimate the value of $g(1.05,2.1,-0.01)$ as appropriate.
   a. $f(x,y) = \dfrac{x^2}{y}$
   b. $g(x,y,z) = x^2yz^2 + \sin yz$
   c. $f(x,y) = xe^y$
   d. $g(x,y,z) = \dfrac{x+y}{z-2y}$

10. Find the unit tangent and unit normal vectors for the curves at the given point. If the magnitude of the tangent and normal vectors are constant, also find the binormal vector.
   a. $\vec{r}(t) = 6\cos t\vec{i} + 2\sin t\vec{j}, t = \dfrac{\pi}{3}$
   d. $\vec{r}(t) = 2\sin t\vec{i} + 2\cos t\vec{j} + 4\vec{k}, P\left(\sqrt{2},\sqrt{2},4\right)$

b. $\vec{r}(t) = \ln t\vec{i} + (t+1)\vec{j}, t = 2$

c. $\vec{r}(t) = 4t\vec{i} - 4t\vec{j} + 2t\vec{k}, t = 2$

e. $\vec{r}(t) = \left(t^3 - 4t\right)\vec{i} + 2t^2\vec{j}, t = 1$

11. Find the equations of the tangent plane and the normal line to the curve at the given point.
    a. $x^2 + 4y^2 + z^2 = 36, P(2,-2,4)$
    b. $y \ln xz^2 = 2, P(e,2,1)$
    c. $2xy - z^3 = 0, P(2,2,2)$
    d. $\vec{r}(u,v) = 2\cos u\vec{i} + v\vec{j} + 2\sin u\vec{k}, P(2,4,0)$
    e. $\vec{r}(u,v) = 3\cos v \cos u\vec{i} + 3\cos v \sin u\vec{j} + 5\sin v\vec{k}, P(3,0,0)$
    f. $x^2 + y^2 + z = 9, P(1,2,4)$
    g. $xyz = 10, P(1,2,5)$
    h. $\vec{r}(u,v) = u\vec{i} + v\vec{j} + uv\vec{k}, P(1,1,1)$
    i. $\vec{r}(u,v) = u\vec{i} + \frac{1}{4}v^3\vec{j} + v\vec{k}, P(-1,2,2)$

12. For the vector-valued function $\vec{r}(t) = t\vec{i} + \sqrt{25-t^2}\,\vec{j} + \sqrt{25-t^2}\,\vec{k}$, find the tangent vector to the graph, at $t = 3$ then use the equations for the line to approximate $\vec{r}(3.1)$.

13. Find the unit tangent vector, the unit normal vector, and the binormal vector for the curve $\vec{r}(t) = \cos(2t)\,\hat{\imath} + \sin(2t)\,\hat{\jmath} + \frac{1}{\pi}t\hat{k}$. Sketch the graph along with the three vectors.

14. Find the equation for the tangent plane for the functions at the given point.
    a. $f(x,y) = 1 + x\ln(xy-5)$, $(2,3)$
    b. $x^2 + xz + 3y^2 = z$, $(1,1,2)$
    c. $\vec{r}(u,v) = \sin u\,\hat{\imath} + \cos u \sin v\,\hat{\jmath} + \sin v\,\hat{k}$, $(1,0,1)$

15. Find the directional derivative of the given function at the point P (if given) in the direction $\vec{v}$. If $\vec{v}$ is not given explicitly, derive it from the second point Q.
    a. $f(x,y) = x^3 - y^3; P(4,3); \vec{v} = \frac{\sqrt{2}}{2}(\vec{i} + \vec{j})$
    b. $f(x,y,z) = xy + yz + xz; P(1,1,1); \vec{v} = 2\vec{i} + \vec{j} - \vec{k}$
    c. $f(x,y) = \frac{y}{x+y}; \vec{v} = \cos\theta\vec{i} + \sin\theta\vec{j}; \theta = -\frac{\pi}{6}$
    d. $f(x,y,z) = xye^z; P(2,4,0); Q(0,0,0)$
    e. $f(x,y) = \arccos xy; P(1,0); \vec{v} = \vec{i} + 5\vec{j}$
    f. $f(x,y) = \sin 2x \cos y; P(0,0); Q\left(\frac{\pi}{2}, \pi\right)$

16. Explain in your own words the difference between the gradient needed for calculating the directional derivative versus the gradient used in calculating the normal vector to a surface.

17. Determine if the surfaces are orthogonal at the curve of intersection Do this by finding the cosine of the angle between the normal/gradient vectors at the point of intersection.
    a. $z = x^2 + y^2, z = 4 - y; (2,-1,5)$
    b. $x^2 + z^2 = 25, y^2 + z^2 = 25; (3,3,4)$

18. Find the directional derivative of the function $f(x, y,) = ye^{-x}$ at the point $(0,4)$ in the direction $\vec{v} = \langle 3,5 \rangle$. What is the direction of the maximum rate of change at the same point?