MTH 277, Homework #9, Spring 2025 Name ____

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) directly on this page if they are short; if too long indicate which page of the work the answer is on and mark it clearly. Use exact values unless specifically asked to round.

1. For each of the following situations, find the unit normal vector to the surface oriented outward and oriented inward. (Label these clearly, both which surface the vector belongs to and what its orientation is.) If the region is defined by more than one surface, find a set of unit normal vector for the outward orientation and the set for the inward orientation.
   1. Surface: $x^2 + y^2 + z^2 = 36$
   2. Surface: $z = 1 - x^2 - y^2, z \geq 0$
   3. Surface: $z = 6 - 3x - 2y$, first octant

   d. Surface: $x^2 + y^2 + z^2 = 36$, first octant

   e. Surface: $z = x^2 + y^2, x^2 + y^2 \leq 4, z = -1$

2. Find the surface area of the given surfaces.
   a. $f(x, y) = 12 + 2x - 3y, R: \{(x, y)|x^2 + y^2 \leq 9\}$
   b. $f(x, y) = 3 + x^{\frac{3}{2}}, R: rectangle\ with\ vertices\ (0,0), (0,4), (3,4), (3,0)$
   c. $f(x, y) = \ln|sec\,x|, R: \left\{(x, y)|0 \leq x \leq \frac{\pi}{4}, 0 \leq y \leq \tan x\right\}$
   d. $f(x, y) = \sqrt{x^2 + y^2}, R: \{(x, y)|x^2 + y^2 \leq 9\}$
   e. $\vec{r}(u, v) = 4u\hat{\imath} - v\hat{\jmath} + v\hat{k}, 0 \leq u \leq 2, 0 \leq v \leq 1$
   f. $\vec{r}(u, v) = 2u\cos v\,\hat{\imath} + 2u\sin v\,\hat{\jmath} + u^2\hat{k}, 0 \leq u \leq 2, 0 \leq v \leq 2\pi$
   g. $\vec{r}(u, v) = \sin u\cos v\,\hat{\imath} + u\hat{\jmath} + \sin u\sin v\,\hat{k}, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$

3. For a function $z = f(x, y)$ and the related function $F(x, y, z)$, state the circumstances in which we use ∇f, and which we use ∇F in each of the circumstances below:
   a. Differentials
   b. Extrema
   c. Tangent planes
   d. How do each of these cases change when working with parametric surfaces?

   e. Gradient fields

   f. Surface area/surface integrals

   g. Directional derivatives

4. Use Stokes' Theorem to evaluate the line integral over the surface.
   a. $\vec{F}(x, y, z) = x^2\hat{\imath} + z^2\hat{\jmath} - xyz\hat{k}, S: z = \sqrt{4 - x^2 - y^2}$
   b. $\vec{F}(x, y, z) = yz\hat{\imath} + (2 - 3y)\hat{\jmath} + (x^2 + y^2)\hat{k}, S: first\ octant\ portion\ of\ x^2 + z^2 = 16\ over\ x^2 + y^2 = 16$
   c. $\vec{F}(x, y, z) = (x + y^2)\hat{\imath} + (y + z^2)\hat{\jmath} + (xy - \sqrt{z})\hat{k}$ over the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ [Hint: construct an equation of the plane containing the triangle.]

5. For the vector field $\vec{F}(x, y, z) = yz\hat{\imath} + 2xz\hat{\jmath} + e^{xy}\hat{k}$ over the region given by the circle $x^2 + y^2 = 16, z = 5$, evaluate the line integral using Stokes' theorem and using the definition of line integral $\oint \vec{F} \cdot d\vec{r}$. Confirm that both methods produce the same solution.

6. For each line integral below, confirm that the vector field is conservative and evaluate it using the Fundamental Theorem of Line Integrals.
   a. $\int_C \vec{F} \cdot d\vec{r}, F(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}, C: line\ \ from\ (0,0,0)\ to\ (5,3,2)$

b. $\int_C (x^2+y^2)dx + 2xydy, \vec{r}_1(t) = t^3\vec{i} + t^2\vec{j}, 0 \le t \le 2; \vec{r}_2(t) = 2\cos t\vec{i} + 2\sin t\,\vec{j}, 0 \le t \le \frac{\pi}{2}$

7. For each of the problems below, describe which line integral methods ***can*** be used to evaluate it, and which one ***should*** be used. Use the best method to evaluate it. Choose from: 1) definition, 2) Fundamental Theorem of Line integrals, 3) Green's Theorem, 4) Stokes' Theorem.
   a. $\int_C xyz^2\, ds$ on the straight line path from $(-1,5,0)$ to $(1,6,4)$.
   b. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = (x+y)\hat{\imath} + (y-z)\hat{\jmath} + z^2\hat{k}, \vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + t^2\hat{k}, 0 \le t \le 1$
   c. $\int_C x^2dx + y^2dy, C: x^2+y^2 = 4$ from $(2,0)$ to $(0,2)$, then on line segment from $(0,2)$ to $(4,3)$.
   d. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = e^x\cos y\,\hat{\imath} + e^x \sin y\,\hat{\jmath}$, on parabola $y = 2x$ from $(-1,2)$ to $(2,8)$
   e. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = xy^2\hat{\imath} + 2x^2y\hat{\jmath}$ where C is the triangle with vertices $(0,0), (2,2), (2,4)$
   f. $\int_C \sin y\, dx + (x\cos y - \sin y)dy$ from $(2,0)$ to $(1,\pi)$
   g. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = \hat{\imath} + (x+yz)\hat{\jmath} + (xy - \sqrt{z})\hat{k}$, whose boundary is the part of the plane $2x + 2y + z = 1$ in the first octant.
   h. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = (e^{-x} + y^2)\hat{\imath} + (e^{-y} + x^2)\hat{\jmath}$ on the curve $y = \cos x$ from, $\left(-\frac{\pi}{2}, 0\right)$ to $\left(\frac{\pi}{2}, 0\right)$ and then the line segment from $\left(\frac{\pi}{2}, 0\right)$ to $\left(-\frac{\pi}{2}, 0\right)$.
   i. $\int_C \vec{F} \cdot d\vec{r}, \vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath} - 2\hat{k}$, where S is the cone $z^2 = x^2 + y^2, 0 \le z \le 4$ oriented downward.

8. Calculate the surface integral $\iint_S f(x,y,z)dS$ of the given function over the given surface. Note: if the surface is parameterized, you will need to make the substitutions for x and y in order to integrate.
   a. $f(x,y,z) = x - 2y + z, S: z = \frac{2}{3}x^{\frac{3}{2}}, 0 \le x \le 1, 0 \le y \le x$
   b. $f(x,y) = xy, S: z = \frac{1}{2}xy, 0 \le x \le 2, 0 \le y \le \sqrt{4 - x^2}$
   c. $f(x,y) = x + y, S: \vec{r}(u,v) = 2\cos u\,\hat{\imath} + 2\sin u\,\hat{\jmath} + v\hat{k}, 0 \le u \le \frac{\pi}{2}, 0 \le v \le 1$
   d. $f(x,y,z) = \sqrt{x^2+y^2+z^2}, S: z = \sqrt{x^2+y^2}, (x-1)^2 + y^2 \le 1$

9. Calculate $\iint_S \vec{F} \cdot \vec{N} dS$ for the given vector field and the given surface.
   a. $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath}, S: z = 6 - 3x - 2y, first\ octant$
   b. $\vec{F}(x,y,z) = x\hat{\imath} + y\hat{\jmath} + z\hat{k}, S: z = 1 - x^2 - y^2, z \le 0$

10. Calculate the flow through the closed surface using one integral per surface and then compare with the Divergence Theorem.
   a. $\vec{F}(x,y,z) = (x+y)\hat{\imath} + y\hat{\jmath} + z\hat{k}, S: z = 16 - x^2 - y^2, z = 0$
   b. $\vec{F}(x,y,z) = 2x\hat{\imath} - 2y\hat{\jmath} + z^2\hat{k}, S: cylinder\ x^2 + y^2 = 4, 0 \le z \le 5$
   c. $\vec{F}(x,y,z) = (2x - y)\hat{\imath} + (z - 2y)\hat{\jmath} + z\hat{k}, S: plane\ 2x + 4y + 2z = 12, z = 0, x = 0, y = 0$
   d. $\vec{F}(x,y,z) = xe^z\hat{\imath} + ye^z\hat{\jmath} + e^z\hat{k}, S: z = 4 = y, z = 0, x = 0, x = 6, y = 0$

11. Explain in your own words the difference between a sink, a source, an incompressible flow. How does this relate to the Divergence Theorem?

12. If a field is conservative and we apply the Divergence Theorem to a closed region under that field, what kind of results should we expect? A sink, a source, an incompressible flow or cannot be determined? Why?

13. Find the area of the surface $y = 4x + z^2$ that lies between the planes $x = 0,\ x = 1,\ z = 0,$ $z = 1$.

14. Find the area of the surface $\vec{r}(u, v) = u^2\hat{\imath} + uv\hat{\jmath} + \frac{1}{2}v^2\hat{k},\ 0 \le u \le 1,\ 0 \le v \le 2$

15. Calculate the flow through the surface using a surface integral or Divergence Theorem as appropriate.
    a. $\int\int_S (x^2z + y^2z)dS$ where S is the hemisphere $x^2 + y^2 + z^2 = 4, z \ge 0$.
    b. $\vec{F}(x, y, z) = (\cos z + xy^2)\hat{\imath} + xe^{-z}\hat{\jmath} + (\sin y + x^2z)\hat{k}$, where S is the surface bounded by the paraboloid $z = x^2 + y^2$, and the plane $z = 4$.